Engendering Students' Experience of Intellectual Need: Is Task Design Enough?

Michael A. Tallman, Oklahoma State University
Jason Martin, University of Central Arkansas
Aaron Weinberg, Ithaca College

The research reported in this paper examines undergraduate students' mathematical thinking while engaged with an intellectual need-provoking (IN-P) task. Our central goal is to infer design characteristics and strategies for the pedagogical implementation of IN-P tasks that orient students to confront the key idea that the task was designed to necessitate. The results of our analysis demonstrate that provoking a cognitive state of intellectual need is nuanced and complex and extends beyond thoughtful task design.

Keywords: Intellectual need, Task design, Perturbation, Student thinking

Engaging with genuine mathematical problems crucial for students' conceptual learning of mathematics. However, much of students' interactions with undergraduate mathematics is "problem-free" in the sense that the mathematical "problems" they encounter are usually solvable by applying skills and knowledge they have already constructed (Fuller, Rabin, & Harel, 2011). These routine procedural exercises do not require students to make strategic decisions or deeply engage with mathematical concepts. While completing such exercises, students often lack a clear mental image of the problem or even an awareness that they are addressing an intellectual challenge. Schoenfeld (e.g., 2016) similarly noted that college mathematics instructors frequently engage students in routine exercises rather than complex problems. In contrast, "problem-laden" activities arise from and are maintained by students' interpretation of a mathematical task that leads them to (1) recognize that their existing knowledge is inadequate to solve the problem, and (2) develop an understanding of what new knowledge they need to construct reason productively about the problem. Well-designed tasks can promote students' awareness of the limitations of their current conceptions and support the development of new mathematical insights.

The positive impact of "desirable difficulties" and "failure-driven scaffolding" on students' mathematics learning is well documented in the literature (e.g., Dewey, 1938; Kapur, 2015; Sinha et al., 2021). For instance, Brousseau (1997) highlighted that engaging with authentic problems can challenge and evolve students' initial ideas. Chevellard's Anthropological Theory of the Didactic suggests that human activities aimed at resolving difficulties are driven by a desire to address problems within institutional settings. Piaget's genetic epistemology also posits that cognitive growth occurs when individuals experience disequilibrium, such as when they encounter unexpected situations. Empirical studies have supported these theoretical claims, demonstrating that students' experience of cognitive incongruities and their engagement in genuine problem-solving prior to participating in formal instruction supports their conceptual learning and development of mathematical reasoning skills.

Consistent with this theoretical tradition of impasse-driven catalysts for cognitive development, Harel and Tall (1991) introduced the concept of *intellectual need*, a notion later formalized by Harel (1998) as the *necessity principle:* "For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need" (p. 501). The principle of intellectual need has been extensively applied in instructional and curriculum design efforts, with empirical research supporting the proposition that students' experience of intellectual need is crucial for their learning. Despite this consensus, however, there remains a lack of practical guidance in the literature on how to assess the extent to which students experience intellectual need and what affordances this psychological state provides for students' mathematical thinking and learning.

The research reported in this paper examines students' mathematical thinking while engaged with an intellectual need-provoking (IN-P) task. We seek to understand the "problem" that students attempted to solve and to document the precise impasses they experienced. Our central goal is to infer design characteristics and strategies for the pedagogical implementation of IN-P tasks that orient students to confront the key idea that the task was designed to necessitate. The results of our analysis demonstrate that provoking a cognitive state of intellectual need is nuanced and complex and extends beyond thoughtful task design.

Theoretical Background

The concept of intellectual need is central to the *DNR-based instruction* framework in mathematics (Harel, 2008), which outlines conditions to provoke students' experience of intellectual need and ensure they internalize and retain what they learn. Harel's *DNR* framework emphasizes that knowledge construction involves experiencing cognitive conflicts or perturbations. Consistent with Piagetian theory, intellectual need arises when students experience cognitive disequilibrium, leading to curiosity that drives them to consider important mathematical ideas and relationships. Unlike mere confusion, intellectual need involves perceiving a situation as problematic and actively working to resolve it.

Intellectual need differs from psychological need and affective need. Affective need relates to a student's initial motivation to solve a problem, driven by personal interests, social obligations, or societal goals. This motivation, along with epistemic emotions like surprise and curiosity, influences students' engagement with mathematical tasks. While affective need and epistemic emotions help initiate and maintain mathematical activity, intellectual need specifically enhances the quality and depth of that activity by driving students to develop powerful mathematical meanings. Unlike other descriptions that view intellectual need primarily as a means to stimulate motivation and engagement, our perspective emphasizes the role of intellectual need in improving the nature of students' mathematical reasoning and learning.

Methods

We engaged mathematics majors and prospective secondary teachers at two public universities in the southern United States in an IN-P task that we modified from Thompson, Carlson, and Silverman (2007) to necessitate a quantitative conception of angle measure[1, 2]. Two students from one university (Group A) and three from another (Group B) collaboratively engaged in the task during class meetings of a mathematics content course for pre-service secondary teachers. The second and third authors were instructors of the same course at two different institutions and coordinated their curriculum to align. We video recorded students' activity and discussion while engaged with the task and collected and scanned and all written products of the students' work. The instructor of Group B did not allow calculators while the students were working on the IN-P task whereas the instructor of Group A did. The students in Group A collaboratively worked on the IN-P task for one 75-minute class; the students in Group B were engaged in the task for one full 50-minute class period and the first 31 minutes of the

[1] We describe what we mean by a "quantitative conception of angle measure" in the section titled, "Conceptual Analysis of the Intellectual Need-Provoking Task."

[2] See Weinberg et al. (2024) for a discussion of the theoretical principles that guided our development of IN-P tasks.

following class meeting. All students had volunteered to participate in the study. The cameras were focused on one primary pair from each class, (Anne and Tessa from Group A and Jane and Eve from Group B). Any other student interacting with the primary pair from either class will simply be referred to as "Student."

The research team collaboratively engaged in grounded theory procedures to analyze the data. We began by developing a provisional set of open codes. Each of the three authors then conducted several iterations of independent analysis of the video data, between which we negotiated our individual interpretations to inform revisions and extensions to our open coding framework. Once we established a stable set of descriptive open codes, each author reanalyzed the videos and the research team again met several times to negotiate differences in our coding and to collaboratively construct an aggregable set of analytical notes that described our justification for applying each code and documented the significance of each coded instance on the students' mathematical thinking and activity. Our subsequent elaboration and refinement of these notes, which involved frequent reengagement with the primary data, resulted in the findings we present below.

Conceptual Analysis of the Intellectual Need-Provoking Task

Students often conceive of angle measure as a quantification of an angle's "openness." Several scholars (e.g., Moore 2010, 2014; Moore et al, 2016; Tallman & Frank, 2020; Thompson, Carlson, & Silverman, 2007) have documented the affordances of understanding an angle's measure as the length of a subtended arc, measured in a unit that varies proportionally with the circumference of the circle centered at the angle's vertex containing the subtended arc. The ratio of the magnitude of the subtended arc and the magnitude of the unit is invariant for an angle with a fixed amount of "openness."

To support students' identification of subtended arc length as the attribute of a geometric object for which angle measures are quantifications, we modified a task from Thompson, Carlson, and Silverman (2007). Our IN-P task simply prompted students to sketch a graph of the function $f(\theta) = \sin(2\cos(\theta))$, with θ measured in radians. Our goal was to problematize students' understanding of the quantity (subtended arc length) represented by the argument of a trigonometric function. Students who conceptualize "$\sin(x)$" as the ratio of the side length of a right triangle opposite the interior angle labeled "x" and the length of the triangle's hypotenuse will likely struggle to interpret this the composite trigonometric function $f(\theta) = \sin(2\cos(\theta))$ in a way that enables them to sketch an accurate graph. As an IN-P task and introduction to a conceptual approach to angle measure and trigonometric functions, we did not expect students to interpret the argument $2\cos(\theta)$ as "twice the horizontal distance of the terminal point to the right of the vertical diameter of a circle centered at the vertex of an angle in standard position." But we did anticipate that students might confront the issue of how the quantity represented by "$2\cos(\theta)$" might be the kind of "thing" (i.e., measurable attribute of an object) that makes sense as the argument of the sine function. We expected that reasoning productively about the task would require understanding the input variable as a representation of the varying values of an angle's measure (as a subtended arc length measured in units of the arc's radius), the argument of f as the length of an arc subtended by a second angle, and "$\sin(2\cos(\theta))$" as the vertical distance of this second angle's terminal point above the horizontal diameter of the circle centered at the vertex of the angle, which is in standard position. See Figure 1 for an illustration of these quantities.

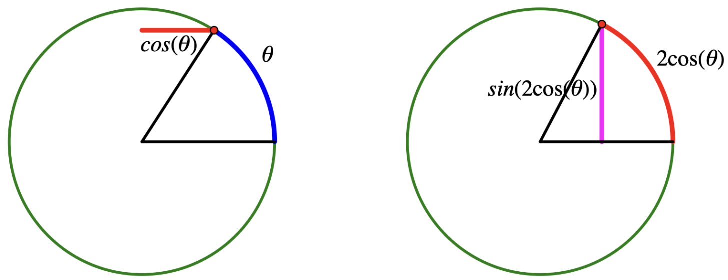

Figure 1. Quantitative interpretation of $f(\theta) = \sin(2\cos(\theta))$.

Results

We present the results of our analysis of two groups of students (Group A and Group B), respectively participating in the study from two different universities. We contrast these students' interpretations of the IN-P task, their goal orientations, and the impasses they encountered to evaluate the extent to which the students' engagement with the task necessitated the meanings we intended to promote.

Group A

The students' initial discussion focused on the graphical transformation contributed by the constant in the expression $\sin(2\cos(\theta))$. They considered whether this constant "stretched" or "compressed" the graph, or if the "2" affects the amplitude (Anne suggested that it might "make sine taller"). They then considered the graphical behavior of $y = \sin(x)$ and $y = 2\cos(x)$ separately and attempted to use sketches of these graphs to reason about the graph of their composition. Anne and Tessa quickly lost confidence in their initial strategy, and transitioned to computing values of $2\cos(\theta)$ for integer multiples of $\pi/4$ in an effort to plot points on a graph. After a long pause following the students' realization that they could not mentally compute $\sin(2\cos(\theta))$ for $\theta = 0$ and $\theta = \pi/4$, the instructor prompted the students to "try to articulate what about this task is difficult and what you are currently struggling with?" The students acknowledged that they were having difficulty with "having cosine inside of sine." Tessa suggested, "cosine and sine are like, they're obviously not inverses of each other, but they're shifted functions. … Then you also have a factor of two, so you're changing that relationship by two. There's just multiple things we have to juggle." Tessa considered how the graph of sine being a shift of $\pi/2$ units to the right of the graph of cosine might influence the graph of f. Anne followed by explaining that while their knowledge of the coordinates of terminal points corresponding to convenient angle measures on the "unit circle" was appropriate for computing values of $2\cos(\theta)$, there were several values of $\sin(\cos(\theta))$ that they were not able to compute using coordinates on the unit circle.

The instructor attempted to direct the students' activity away from their effort to plot points and instead to infer the behavior of the graph that connects these points. He suggested, "Is there a way to think about how output values are related to input values as θ varies?" Tessa was confident that the graph of f "is going to oscillate because of what we know based off sine and cosine. Um, the bounds at which it is oscillating between … that's the part that we would have to compute more." Tessa expressed the need to compute enough values of $f(\theta)$ for select values of θ to infer the graphical behavior of f between these input values. She expected that these computations might reveal "the bounds at which it is oscillating between." Anne expressed that she thought about the problem similarly to how Tessa described her reasoning.

In response to being asked to propose a question they would want answered to help them progress towards a solution, Tessa suggested, “Confirmation on basic properties and rules for layering [i.e., composing] those things.” Tessa was oriented to assume that there was some algebraic property or rule for composing trigonometric functions presented in a prior course that she was currently unable to recall. In contrast, Anne expressed that she wanted an answer to the question, “When you take a radian measure and put it through a trig function, will you get a radian measure out still? That lets me reason if what I’m putting into sine makes sense.”

Anne’s awareness of the need to associate a quantity (i.e., measurable attribute of an object) to the expression “2cos(θ)” so that she might be able to reason about the sine value of this argument penetrates a key issue in developing a productive meaning for angle measure: How can the quantities for which sin(x) and cos(x) are respective measures be the same kind of quantity represented by the input x? Anne pursued this reflection and proposed the additional query, “The thing is when you’re graphing these, I realized that the 1 and –1 that you’re oscillating between, is that a radian measure? The y-value that we get out after putting in an x-value?” After considering her question, Anne concluded, “Whatever we’re putting into sine doesn’t make sense to put into sine.” Tessa agreed and attempted to articulate a justification for Anna’s conclusion:

> We were talking about when you do cos(0) and you get 1 as the output, that 1 corresponds to your length of your radius of your circle, like when you’re thinking unit circle, and so you wouldn’t plug a radius unit back into sine. You need to plug in radians, so you need to convert that.

Tessa’s agreement with and elaboration of Anna’s comment demonstrates their initial lack of association between quantities represented by sin(θ) and cos(θ) being measured in units of the “length of the *radius* of your circle” and the quantity represented by an angle’s measure in *radians*. Still, that the students interpreted the task in a way that guided them to consider the quantities represented by the inputs and outputs of trigonometric functions has likely implications for their orientations towards and expectations for learning about angle measure in subsequent lessons.

Group B

After being given the prompt to sketch the graph of $f(\theta) = \sin(2\cos(\theta))$, both Jane and Eve began to evaluate the function at the angle measures for which they knew the value of cos(θ) (see Figure 2). Both Jane and Eve’s scheme for graphing functions included plotting points based on evaluating functions for certain input values. This was demonstrated by both Jane and Eve as they would later add axes and plot points based on the values they computed. Their graphing of trigonometric functions additionally included selecting integer multiples of $\pi/2$ to produce readily known values for the argument of f.

$\cos\theta$		$2\cos(\theta)$	$\sin(2\cos(\theta))$
$\cos(0) = 1$		$2\cos(0) = 2$	$\sin(2\cos(0)) =$ $\sin(2) =$ ___
$\cos(\frac{\pi}{2}) = 0$		$2\cos(\frac{\pi}{2}) = 0$	$\sin(2\cos(\frac{\pi}{2})) =$ $\sin(0) = 0$
$\cos(\pi) = -1$	x2	$2\cos(\pi) = -2$	$\sin(2\cos(\pi)) =$ $\sin(-2) =$ ___
$\cos(\frac{3\pi}{2}) = 0$		$2\cos(\frac{3\pi}{2}) = 0$	$\sin(2\cos(\frac{3\pi}{2})) = 0$ $\sin(0) = 0$
$\cos(2\pi) = 1$		$2\cos(2\pi) = 2$	

Figure 2. Eve’s notes for evaluating cos(θ) *for convenient angle measures and how known values from* cos(θ) *can be used to evaluate* sin(2cos(θ)).

After about five minutes of working, Eve observed that she could evaluate $f(\theta) = \sin(2\cos(\theta))$ for $\theta = {}^{\pi}/_{2}$ and $\theta = {}^{3\pi}/_{2}$ (see the equations marked with asterisks in Figure 2), but was unable to evaluate $f(\theta) = \sin(2\cos(\theta))$ for the input values 0, π, and 2π because she was unable to determine exact values for sin(2) and sin(–2). Since the function f is a composition of two trigonometric functions, values of $2\cos(\theta)$ are often inconvenient as the argument into $\sin(\theta)$ if a student is looking for known values from "the unit circle."

For the equations corresponding to sin(2) and sin(–2), Eve then added the blank line to the right of the equality to indicate not knowing these values, later coming back to highlight these equations to reemphasize how angles convenient for evaluating $\cos(\theta)$ did not reliably produce known values for $\sin(2\cos(\theta))$. When comparing their work, Jane described her approach and admitted that she too was unable to evaluate f for particular values of θ. "And I'm still running into that problem, where I don't know the sine of these values [pointing to her table of values that contained a column for $\sin(2\cos(\theta))$ that was empty except for when $\theta = 0$]." Jane and Eve problematized their situation as follows:

Jane: Yeah. Well not, like, sin(2).
Eve: Like, sin(2), I don't know that. [...]
Jane: [Looking at Eve's notes] So basically, all we're getting is the pi-over-twos [likely in reference to ${}^{\pi}/_{2}$ and ${}^{3\pi}/_{2}$].
Eve: Basically. But I don't know what it does in between.

At this moment, Jane and Eve had encountered an impasse that their activated cognitive structures were insufficient to resolve. This impasse included their recognized inability to produce values for the output of $y = f(\theta)$ for no more than a few select values of the input. Because of this, they acknowledged not knowing the behavior of the function between these select few values of θ. The students then proceeded to locate additional points and infer the graphical shape between the points based on contextual clues. For example, Jane indicated searching for the y-intercept. Both students confused locating the y-intercept with solving $\sin(2\cos(\theta)) = 0$ for θ to eventually get $\theta = {}^{\pi}/_{2}$. Eve noted how multiple values for the input produced identical output values, such as $\theta = \pi$ and $\theta = -\pi$, both of which correspond to an output value equivalent to sin(–2). Eve then leveraged her knowledge of the basic shapes of the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ to produce the graph of $y = f(\theta)$ displayed in Figure 3.

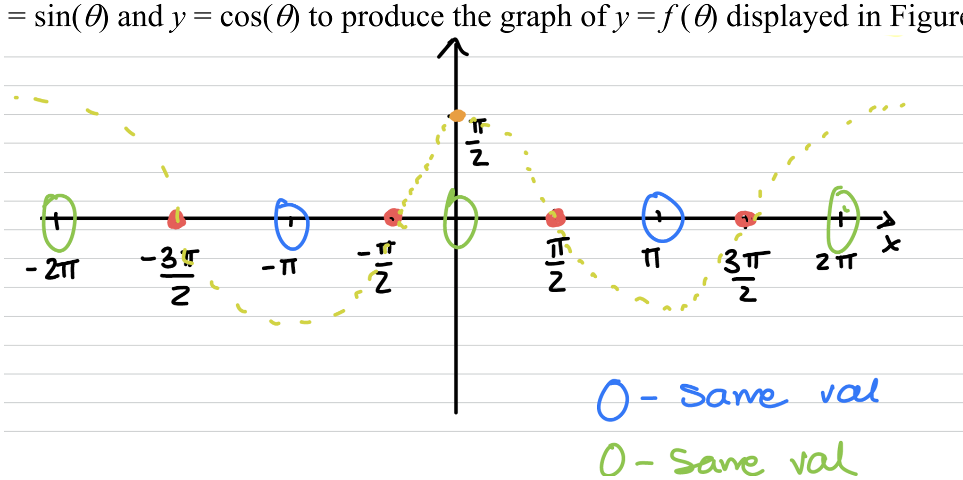

Figure 3. Eve's initial graph for $f(\theta) = \sin(2\cos(\theta))$ *where she notes where particular values of input would correspond to the same value of output.*

Jane momentarily proposed an alternative graph that did not include negative output values while Eve indicated thinking about “regular sine and cosine graph.” They concluded that the graph of $y = f(\theta)$ must “go under” the horizontal axis because “both” graphs for $y = \sin(\theta)$ and $y = \cos(\theta)$ “go under” the horizontal axis. Other attributes of trigonometric functions were also mentioned by the pair, such as period and amplitude. Following this, Eve declared that “We have a sketch.” She went on to acknowledge that she still wanted to know the value for sin(2) and indicated that if she knew this value, then she would know if her graph was accurate.

Notably absent from Eve and Jane’s expressed reasoning to this moment were indications of them coordinating the quantities represented by the input, argument, and output of f. Although the students labeled locations on the unit circle and evaluated functions for particular values of θ, they did not indicate, for example, conceiving of $\cos(\theta)$ as a measurement of horizontal displacement of the terminal point from the vertical diameter of a circle centered at the angle’s vertex. Nor did they indicate that via the function composition, the argument of the sine function $2\cos(\theta)$ could be conceived of as the directed length of a subtended arc that is twice the length of the aforementioned horizontal displacement. And, most related to the students’ experience of intellectual need, neither Eve nor Jane appeared to confront the issue of how the output of one trigonometric function could be the kind of thing that can serve as the input into another trigonometric function. Although these students had encountered an impasse and eventually produced a graph for which they were confident represented the covariational relationship between θ and $\sin(2\cos(\theta))$, they had not yet confronted the central ideas rooted in quantitative reasoning about angle measure that we had designed the task to necessitate.

Discussion

Our study reveals significant variations in how individual students interpreted and engaged with the intellectual need-provoking (IN-P) task. These differences stem from students’ diverse assumptions about the prior knowledge and mathematical meanings that must be applied to successfully complete the task. Our findings demonstrate that tasks alone have limited effectiveness in inducing a cognitive state of intellectual need. Instead, achieving the desired cognitive engagement requires careful pedagogical strategies and a deep understanding of the students’ pre-existing conceptions and expectations. This study highlights that fostering intellectual need is more intricate and nuanced process than previously suggested in the literature.

The findings from the study suggest several important implications for educational practice. First, simply implementing intellectual need-provoking (IN-P) tasks is not sufficient. Educators need to carefully design tasks to ensure they appropriately connect with students’ prior knowledge. Instructors also need to consider the pedagogical actions that might challenge students’ thinking about the core mathematical ideas reflected in the IN-P task. This may involve providing in-the-moment scaffolding of tasks to orient students’ attention to the central mathematical issue(s) that the task was designed to necessitate, and to do so in ways that are sensitive to the strategies a student has already employed, the operational goal the student is working to accomplish, and the mathematical facts, skills, and meanings the student brings to the task. Effective teaching requires flexibility and responsiveness. Educators should be prepared to modify their pedagogical strategies based on their ongoing assessment of students’ engagement and comprehension. This may include adjusting the level of difficulty of tasks, providing additional explanations, or incorporating different teaching methods to better support students’ conceptual learning of mathematics.

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