New Methods for Analyzing Students' Reasoning About Definite Integrals in Introductory Physics

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Over the past decade, research on students' understanding of integrals has grown significantly. While previous studies have noted students' difficulties in coordinating the "product layer" with definite integrals in applied contexts, they often overlook the diverse knowledge resources students use in their moment-to-moment reasoning. Additionally, effective methods for analyzing this complexity have been lacking. This paper introduces new methods for examining students' reasoning about the "product layer", $f(x) \cdot dx$*, and definite integrals, combining a modified dynamic conceptual blending diagram and an expanded Visualization/Analysis/Physics (VAP)-model. We demonstrate these methods with an example of an integration task, illustrating their application and impact on integration research (see Sus & Izsák, 2024b for further examples). This contribution advances the literature on integration research by providing a more nuanced analysis of students' reasoning.*

Keywords: Integration, Product Layer, Knowledge-in-Pieces, Expanded VAP-Model, Dynamic Conceptual Blending

Introduction and Literature Review

Integration is fundamental in modeling many applied phenomena and is a core topic in calculus. It is a complex concept, structured around the coordination of multiple quantities through various mathematical operations. A significant body of literature has explored students' understanding of integrals in applied contexts (e.g., Meredith & Marrongelle, 2008; Cui et al., 2006). Researchers have identified challenges students face when modeling quantities like distance and work using rectangular areas (e.g., Thompson et al., 2013; Nguyen & Rebello, 2011; Christensen & Thompson, 2010).

Students often interpret the definite integral in one of three ways: as the "area under the curve," as a "derivative/antiderivative" relationship, or as "adding up pieces through multiplicatively-based summation" (e.g., Jones, 2015a, 2015b; Ely & Jones, 2023; Jones & Ely, 2023). Research suggests that the third is especially useful when applying integrals to physics, engineering, and other fields. However, there is ongoing debate about the range of situations where the multiplicatively-based summation and Riemann sum interpretations are applicable. Meredith and Marrongelle (2008) argued that using the product structure $f(x) \cdot \Delta x$ in a Riemann sum is sufficient for modeling accumulation only when the integrand represents a rate of change or density. However, many contexts do not naturally conform to this quantitative structure.

Particular challenges arise when reasoning about the "product layer" in definite integrals, as highlighted by several studies (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). For example, Sealey (2014) developed a framework for understanding Riemann sums and definite integrals, identifying the "product layer" as a particularly complex and challenging concept for students.

This paper builds on Oehrtman and Simmons' (2023) discussion of Emergent Quantitative Models for definite integrals, which includes three primary models: basic, local, and global. The researchers provided examples showing how student reasoning can involve varied interactions among these models, such as choosing which factor in a product to treat as infinitesimal and how to approach partitioning. However, they stopped short of exploring knowledge resources which students applied while coordinating multiplication with definite integrals through the construction of all three models. This omission is significant in being able to analyze the complexity of students' reasoning about multiplication and definite integrals in applied situations.

In other studies (Sus & Izsák, 2024a; Izsák & Sus, 2024), researchers have examined how students coordinate multiplication with definite integrals, finding that students draw on a variety of mathematical and physics knowledge resources, their combinations, as well as related drawings. These resources fluctuate depending on the problem context, which can lead to shifts in student reasoning. Such instances reveal the complexity of students' reasoning about the "product layer," highlighting the need for a closer examination of students' moment-to-moment reasoning and the knowledge resources they employ.

In this paper, we propose new methods for analyzing the complexity of students' moment-to-moment reasoning about the "product layer" and definite integrals with a focus on the knowledge resources they use. This serves as a significant contribution to the existing literature on integration.

Theoretical Perspective

In this section, we outline the theoretical perspective we use to analyze students' complex reasoning about mathematical concepts, specifically multiplicative structures in applied integration contexts. This perspective includes three components: the knowledge-in-pieces epistemology, the dynamic conceptual blending framework (e.g., Van den Eynde et al., 2020), and visual conceptual blending (Cunha et al., 2020).

The Knowledge-in-Pieces Epistemological Perspective

The knowledge-in-pieces epistemological perspective was first developed in science education research on conceptual change (e.g., diSessa, 1993, 2006). It has since been applied to various topics in mathematics, including whole-number multiplication (Izsák, 2005), functions (e.g., Izsák, 2004; Moschkovich, 1998), the law of large numbers (Wagner, 2006), and integrals (Jones, 2013). From this perspective reasoning is supported by diverse, fine-grained knowledge resources and more novice knowledge evolves into more expert knowledge through processes such as the construction of knowledge resources that are sensitive to context for activation, refinement of contexts in which resources are applied, and reorganization that can involve forming new connections among some recourses and loosing connections among others. In the present study, we are particularly interested in knowledge resources students evidenced when coordinating multiplication with definite integrals in the introductory physics context.

Conceptual Blending

Original version of the conceptual blending perspective. Conceptual blending was originally introduced by Fauconnier and Turner (1998) to model how people create new meaning in linguistic contexts by selectively combining information from different domains of previous experience. Figure 1 shows a schematic representation of the conceptual blending diagram. In its basic form, the framework consists of four connected mental spaces: two partially matched input spaces, a generic space, and the blended space. Input spaces are small self-contained regions of conceptual ideas (knowledge resources). The generic space provides the underlying structure to the input spaces, identifying commonalities in content and structure. The blended space is constructed through the combination of selected elements from the input spaces. Fauconnier and Turner distinguished three approaches to form a blended space: composition (blending can compose elements from the input spaces to provide relations that do not exist in the separate input spaces), completion (adding new elements based on background models that are brought into the blend unconsciously), and elaboration (treating the blend as a simulation and "running" it imaginatively, which creates new insights). Also, the blend has emergent dynamics: It can be "modified" while its connections to the other spaces remain in place.

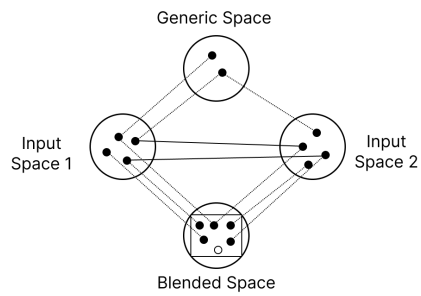


Figure 1. Schematic representation of a blending diagram (Fauconnier & Turner, 1998, p.143).

For example, Fauconnier and Turner (2002) illustrated the idea of a blend using the construct of a "computer virus" which blends meaning from the two domains of "computers" (files, software) and "viruses" (harmful, discrete, replicating) to create a new term that describes a harmful, replicating piece of software (Fauconnier & Turner, 2002, pp. 274–275).

Adapted conceptual blending perspective: Students' use of mathematics in physics. Prior research studies (e.g., Bing & Redish, 2007; Hu & Rebello, 2013; Odden, 2021; Wittmann, 2010) have investigated conceptual blending to describe students' use of mathematics in physics. For instance, Bing and Redish were the first to introduce the language of conceptual blending as a way to analyze problem solving in physics at the introductory level. Schermerhorn and Thompson (2014) used the symbolic forms framework (Sherin, 2001) and incorporated it into the conceptual blending framework to describe students' construction of differential length vectors with extensions to equation construction and interpretation generally.

Hu and Rebello (2013) adapted the conceptual blending framework from cognitive science to make sense of ways in which students combined their knowledge from calculus and physics to set up integrals. They found that many students were not able to blend their mathematics and physics knowledge in a productive way even though they had the required mathematics knowledge. An example of their blending diagram consists of three input spaces, symbolic ($\rho(x), R = \frac{\rho L}{A}$), math notion space (function, integral as a sum, '$d[\,]$' as a variable of integration), physics space (resistivity, resistance, length), and the blended space (resistivity varies along length, $\frac{\rho(x)L}{A}$ substitute resistivity, $\int_0^L \frac{\rho(x)L}{A} dx$ sum up over).

Van den Eynde et al. (2020) utilized the conceptual blending framework to study student thinking in integrating mathematics and physics within the context of the heat equation. They introduced dynamic blending, enhancing earlier diagrams by incorporating time stamps and both implicit and explicit connections to better capture the dynamics of reasoning.

Odden (2021) investigated how undergraduate physics students create conceptual connections to overcome challenges with complex and counterintuitive ideas in introductory physics. The study demonstrated how students construct and use conceptual blends to understand difficult physics concepts.

In this paper, we take a dynamic conceptual blending perspective (Van den Eynde et al., 2020) which incorporates the dynamic aspect of students' reasoning and explicitly visualizes how students switch between symbolic, mathematics, physics, and blended elements throughout their reasoning process.

Visual conceptual blending. Visual conceptual blended elements emerge from a process of visual blending grounded in conceptual understanding. Visual blending involves merging two or more visual representations (such as images) to create new ones. When visual and conceptual blending are combined, they result in what is known as visual conceptual blending (Cunha et al., 2018). This blending process plays a crucial role in students' moment-to-moment reasoning, supporting the creation of new visual elements that reflect their dynamic thought processes.

New Methods

In this paper, we use a modified version of the dynamic conceptual blending diagram from Van den Eynde et al. (2020) to explore students' reasoning about multiplicative structures and their application to integration tasks in physics. We introduce a new step for identifying activated and deactivated elements and connections and apply a different coding scheme to elements in the transcript. Our modified dynamic conceptual blending diagram (MDCBD) is illustrated in Fig. 2.

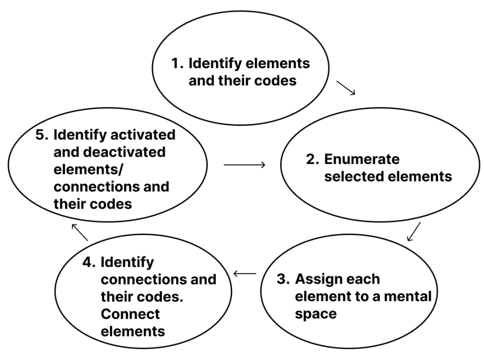


Figure 2. Schematic representation of MDCBD.

Step 1. We identify elements from verbatim transcript, which represent pieces of knowledge or resources. These elements are defined as new steps, ideas, concepts, or phrases in a student's reasoning. Unlike Van den Eynde et al. (2020), we recorded all elements from the transcript and categorized them into two groups: more confident and more tentative elements. An element is considered more confident if a student expresses confidence in introducing it, using phrases such as "I remember," "I am sure," or "I recall." Conversely, elements are classified as more tentative if the student shows hesitation, using phrases like "I guess" or "I am not sure." More confident elements are coded with solid dots, while more tentative elements are marked with empty dots. Elements introduced in the problem statement are also included in the MDCBD and can be either more confident or more tentative. Notably, we do not categorize elements as correct or incorrect within the MDCBD.

Step 2. We enumerate only those elements that capture students' reasoning about the multiplicative structure and its coordination with the concept of definite integrals. The elements are listed in the order they appeared in the transcript, allowing us to better understand the dynamics of students' reasoning throughout the interview.

Step 3. We assign each enumerated element to one of four mental spaces in the MDCBD: three input spaces (symbolic, physics, and mathematics) and one blended space. The *symbolic* *space* includes mathematical and physical symbols and notations. The *physics space* encompasses elements related to physics knowledge (e.g., concepts, statements, and drawings about physics laws) without introducing new mathematical input. The *mathematics space* includes elements related to mathematical knowledge (e.g., drawings, functions, equations, derivatives, and integrals) without new physical input. The *blended space* consists of elements that integrate ideas from the symbolic, mathematics, and physics spaces, including equations and drawings. We denote enumerated elements assigned to each mental space by S1, M1, B1, P1, etc. In our study, we use the generic space to represent the overall structure based on inferred reasoning, as interpreted by the researchers. Since our analysis focuses on students' statements and reasoning, we exclude the generic space from our diagram.

Step 4. We trace students' dynamic reasoning about multiplicative structures by connecting the selected elements across all spaces and within each space. These connections are categorized into two groups: more confident and more tentative. These connections are defined similarly to the more confident and more tentative elements described in Step 1. To represent these connections visually, we use solid green lines for more confident connections and dotted green lines for more tentative connections.

Step 5. In contrast to the dynamic blending framework described by Van den Eynde et al. (2020), our approach accounts for dynamic interactions among elements and their modifications. Modifications refer to students' actions of activating or deactivating elements and connections, whether more confident or more tentative. Activation occurs when a student recognizes the relevance of an element to the current phenomena. Activated elements are represented by blue squared boxes, while deactivated elements are represented by red squared boxes. At this stage, we identify elements modified by students and repeated steps 2 to 4 for the updated set of elements. Consequently, a connection identified in Step 4 may initially be active and later become inactive. An activated connection indicates that a student has recognized the relevance of elements to the current phenomena. Activated explicit connections are represented by solid blue lines, while deactivated explicit connections are shown with solid red lines. Activated and deactivated implicit connections are denoted by dotted blue and red lines, respectively.

Figure 3a provides a schematic representation of the modified dynamic conceptual blending diagram, including all the coding conventions discussed. By elaborating on the four mental spaces, the connections among elements within those spaces, and their dynamic changes, we described students' moment-to-moment reasoning about the "product layer" when blending ideas from the symbolic, mathematical, and physical input spaces.

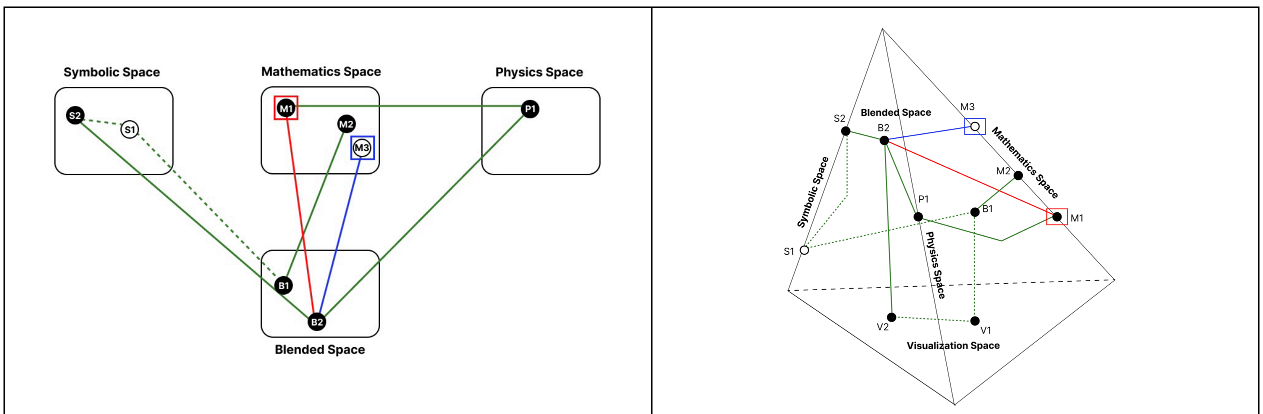
The Expanded VAP-Model

Zimmerman (1991) wrote that "visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p.136). Other researchers contend that in order for students to construct a conceptual understanding of mathematical concepts, both visual and analytic reasoning must be present and integrated (Aspinwall & Shaw, 2002; Zazkis et al., 1996). There is also a literature dedicated to graph interpretation and construction outside of mathematics. For instance, Beichner (1994) developed a test to determine students' interpretations of kinematics graphs. Therefore, we consider visual representations (drawings) to be an important part of a student's moment-to-moment reasoning when connecting mathematics (calculus) to physics.

Our perspective extends the VAP-model which Zazkis (2013) developed to analyze students' visual (V), analytical (A), and physical (P) modes of reasoning, and the transitions between reasoning with graphs, functions, and physical concepts. The VAP-model diagram has a tetrahedron shape where its edges represent three modes of reasoning. The transition between the modes is a disorderly, unpredictable path that moves upward with time and has a spiral form. The upward direction indicates that visualization and analysis become closer to each other as one moves to more advance levels.

In our extended model, we incorporate five reasoning spaces: symbolic (S), mathematics (M), physics (P), blended (B), and visualization (V). The spaces S, M, and P are represented by the vertical edges of a tetrahedron, with the blended space occupying the interior. The altitude of each element within the tetrahedron reflects its activation time, moving from bottom to top. The coding of elements and their connections in spaces S, M, P, and B follows the same scheme as the modified dynamic conceptual blending diagram (MDCBD) described earlier.

Under visualization space V we consider a visual blended space complemented by a conceptual layer developed through elaboration (e.g., Cunha et al., 2020). It is represented by elements located at the base of the tetrahedron, arranged chronologically (V1, V2, etc.). These elements, drawings, are connected to the blended elements in space B, with coding consistent across all spaces. Figure 3b provides a schematic representation of the expanded VAP-model diagram.



(a) (b)

Figure 3. The MDCBD and the expanded VAP-model diagram.

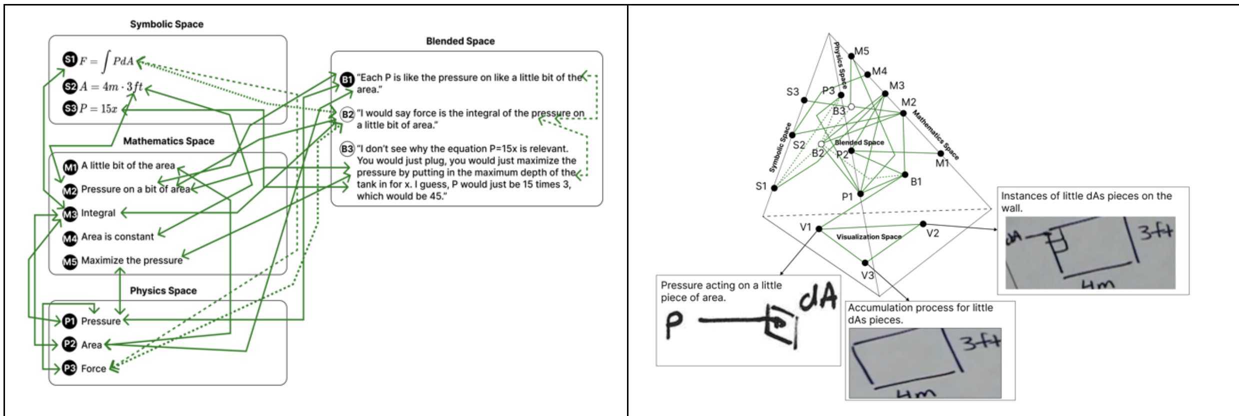
Water Pressure Task: An Illustrative Example

In this section, we provide an illustrative example of how the above methods can be applied to analyze student's moment-to-moment dynamic reasoning about the "product layer" and the definite integral in an applied context.

The example comes from a college student, Matthew, enrolled in a second semester calculus-based physics course. Matthew was working on a Water Pressure Task (adapted from Sealey, 2014) during a one-to-one interview session. The task asked him to estimate the total force from the water pressure on a $4m \cdot 3ft$ vertical side wall of a tank, where the pressure was given by the function $P = 15x$ and x was the depth of the water. Here is a sample of interview data.

Matthew began by writing down the given data which led him to think that "Each P is like the pressure on like a little bit of the area." He claimed that he did not know the actual relation between the quantities in the given problem situation: "I don't know the actual relation, so I'm just guessing." Then he stated: "I would say force is the integral of the pressure on a little bit of area [$F = \int PdA$]. And umm the area is, well the area is constant essentially makes sense [$A = 4m \cdot 3ft$]." Matthew expressed some confusion about the depth of the water to be given as x, where the tank is full of water. As a result, he stated that he did not see why the equation $P = 15x$ is relevant because "you would just plug, you would just maximize the pressure by putting in the maximum depth of the tank in for x." He continued: "So, I guess P would be 15 times 3, which would be 45." Later, Matthew added the graphical representation showing how the pressure is acting on a little bit of area (see drawings in Figure 4b).

We now illustrate (see Fig. 4) how this piece of data can be analyzed using MDCBD and the expanded VAP-model diagram given in Figure 3.



(a)	(b)

Figure 4. The MDCBD and the expanded VAP-model diagram.

Discussion and Conclusion

Modeling Matthew's responses with the modified dynamic conceptual blending diagram (Figure 4a) and the expanded VAP-model diagram (Figure 4b) allowed us to conduct a fine-grained analysis of his reasoning in the given problem situation. We can see that Matthew applied knowledge resources coming from three input spaces to generate blending elements such as B1, B2, and B3. For instance, combing elements S1, M1, M2, M3, P1, P2 and P3, the student generated a blended element B2, which stood for the force definition via the integral representation and represented the "product layer" structured as PdA. Later, the student added a visual representation of the pressure acting on a little bit of area, which was captured by the element V1 of the visualization space and corresponded to the expression PdA. He demonstrated instances of little bits of area by V2 and blending it with the element V1 produced the accumulation process given by V3.

Overall, creating these diagrams enables researchers to move beyond analyzing word choices, gestures, or inscriptions, allowing for a deeper exploration of the meanings that students construct. For instance, the diagrams in Figure 4 helped us capture the complexity of Matthew's dynamic reasoning about the definite integral by identifying a set of knowledge resources and their interactions. Analyzing these diagrams alongside the data can help researchers refine their theories of students' mathematical understanding in applied contexts and significantly contribute to the existing literature on integration.

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