The Complexity Coordinating the “Product Layer” and Definite Integrals in Introductory Physics

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Research on students’ construction and interpretation of integrals has grown significantly over the past decade. The present paper contributes to recent research on challenges which students experience while coordinating the “product layer”, $f(x) \cdot dx$, and the definite integral in applied contexts. In particular, we demonstrate a case study which examines how an undergraduate student, Matthew, linked the “product layer” and definite integral in an introductory physics task by blending mathematics, physics, and visualization knowledge resources. We analyzed Matthew’s moment-to-moment reasoning by utilizing a modified dynamic conceptual blending diagram (MDCBD), coming from the dynamic conceptual blending perspective, and an expanded Visualization/Analysis/Physics (VAP)-model, as detailed in (Sus & Izsák, 2024). Our findings indicate that students’ understanding of multiplication with quantities in applied integration contexts is highly complex and that neglecting the dynamic interactions of knowledge resources represents a significant gap in integration research.

Keywords: Integration, Product Layer, Knowledge-in-Pieces, Expanded VAP-Model, Dynamic Conceptual Blending

Introduction and Literature Review

Students often struggle to understand and apply the concept of definite integrals in applied contexts, particularly when reasoning about the “product layer,” which involves interpreting the integral as a multiplicatively-based summation of quantities (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). These difficulties become apparent when students attempt to model quantities such as distance and work using rectangular areas or other visual representations (e.g., Thompson et al., 2013; Nguyen & Rebello, 2011). While past research has explored students’ general understanding of integrals (e.g., Meredith & Marrongelle, 2008; Cui et al., 2006; Yeatts & Hundhausen, 1992), less attention has been given to how they reason about this specific “product layer” within integrals.

Sealey (2014) identified the “product layer” as a complex area for students, and Oehrtman and Simmons (2023) analyzed how students construct definite integrals in applied problems, identifying three emergent quantitative models: basic, local, and global. However, their work did not examine closely how students’ understanding of multiplication (in particular, knowledge resources) influences their construction of local models, or how their use of drawings affects their reasoning.

Further research by Izsák and Sus (2024) shows that students rely on various mathematical and physics knowledge resources when coordinating multiplication with definite integrals, and that these resources fluctuate based on the problem context. Visual representations, as noted by Berry and Nyman (2003), play a crucial role in guiding students’ reasoning about definite integrals and are essential for reasoning about the “product layer.” Research consistently suggests that integrating visual and analytical reasoning is key to fostering conceptual understanding in mathematics (Aspinwall & Shaw, 2002; Zazkis et al., 1996).

Our study aims to address these gaps by combining the approaches of Izsák and Sus (2024) with a focus on visual reasoning. We examine how students apply knowledge resources, including drawings, when coordinating multiplication with definite integrals in applied contexts.

To explore this, we use a novel methodological approach that integrates conceptual blending theory and visualization theory (Sus & Izsák, 2024). Our research questions are: *What knowledge resources do students use in their moment-to-moment reasoning about the "product layer" construction in introductory physics? How do students apply drawings in their moment-to-moment reasoning about the "product layer" construction in introductory physics?*

Theoretical Perspectives and Methodological Approach

In this section, we present theoretical perspectives and a new methodological approach which best spans the results obtained in the study. It includes three components — the knowledge-in-pieces cognitive perspective, the modified dynamic conceptual blending diagram, coming from the dynamic conceptual blending perspective (Van den Eynde et al., 2020), and the expanded VAP-model diagram (Zazkis, 2013) related to the visual conceptual blending perspective (Cunha et al., 2020).

The Knowledge-in-Pieces Epistemological Perspective

The perspective was first developed in science education research on conceptual change (diSessa, 1993, 2006) and has since been applied to topics in mathematics, such as whole-number multiplication (Izsák, 2005), functions (Izsák, 2004; Moschkovich, 1998), and integrals (Jones, 2013). It posits that reasoning is supported by diverse, fine-grained knowledge resources, which evolve from novice to expert through the construction, refinement, and reorganization of these resources. In this study, we focused on the knowledge resources students use when coordinating multiplication with a definite integral in introductory physics.

Dynamic Conceptual Blending, Modified Dynamic Conceptual Blending Diagram

Fauconnier and Turner (1998) introduced conceptual blending to model how people create new meaning by combining information from different experiential domains. Studies (e.g., Bing & Redish, 2007; Hu & Rebello, 2013; Wittmann, 2010) have applied this framework to describe students' use of mathematics in physics. In this study, we take the conceptual blending perspective (Van den Eynde et al., 2020) to analyze students' dynamic reasoning about the "product layer" in applied integration. We construct the modified dynamic conceptual blending diagram (MDCBD) through the following steps: (1) identifying and coding elements (knowledge resources); (2) listing selected elements chronologically; (3) assigning them to mental spaces — mathematics (mathematical concepts, including drawings and equations), physics (physical concepts and quantities, including drawings), symbolic (technical roles of mathematics), and blended (combinations of symbolic, mathematical, and physical ideas, including equations and drawings) and denoting them by M1, S1, P1, B1, etc.; (4) connecting selected elements with coded links; and (5) coding activated/deactivated elements and revisiting step (2).

Elements and connections are coded as more confident (e.g., "I remember") or more tentative (e.g., "I guess"), with more confident elements/connections shown as solid dots/lines and more tentative as empty dots/dotted lines. Activated/deactivated elements are marked with squared blue/red boxes, while connections are marked with blue/red solid or dashed lines, respectively. Figure 6a illustrates the MDCBD and its coding scheme.

By analyzing the elements and connections of these mental spaces, and their dynamic changes, we captured students' moment-to-moment reasoning about the "product layer" as they blended ideas from symbolic, mathematical, and physical input spaces.

The Expanded VAP-Model

Our approach builds on the VAP-model (Zazkis, 2013), which examines students' transitions between visual (V), analytical (A), and physical (P) reasoning modes in relation to graphs, functions, and physical concepts. We extended this model by incorporating five reasoning spaces: symbolic (S), mathematical (M), physical (P), blended (B), and visualization (V).

In our expanded model, the spaces S, M, and P are represented by the vertical edges of a tetrahedron, with the blended space occupying the interior. The altitude of each element within the tetrahedron reflects its activation time, moving from bottom to top. The coding of elements and their connections in spaces S, M, P, and B follows the same scheme as the modified dynamic conceptual blending diagram (MDCBD) described earlier.

Under visualization space V we consider a visual blended space complemented by a conceptual layer developed through elaboration (e.g., Cunha et al., 2020). It is represented by elements located at the base of the tetrahedron, arranged chronologically (V1, V2, etc.). These drawing elements are connected to the blended elements in space B, with coding consistent across all spaces. Figure 6b provides a schematic representation of the expanded VAP-model diagram.

Data

We conducted three one-hour, one-on-one, semi-structured think-aloud interviews (e.g., Bernard, 1994; Ginsburg, 1997) with three students enrolled in a second-semester calculus-based physics course at a selective university. The goal was to assess how these students reasoned about the integral concept in various problem situations, focusing on whether and how they constructed a "product layer."

The first interview explored students' reasoning about multiplication in four applied problem situations, none of which mentioned integrals but required multiplication followed by addition to construct a total. The second and third interviews focused on connecting definite integrals to three and four applied problem situations, respectively. We recorded the interviews with two cameras — one capturing the student and interviewer, and the other capturing the student's written work. After synchronizing the videos and transcribing the interviews verbatim, we reviewed them side-by-side with the transcripts to ensure accuracy. Throughout, we closely examined the knowledge resources students used in building the "product layer."

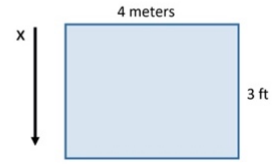
All students struggled at times to coordinate multiplication with the "product layer" in different problem situations. Matthew, who self-reported strong calculus skills (Calculus BC, 5), stood out. In the second interview (Water Pressure task), he demonstrated two types of reasoning about the "product layer," initially arriving at a normatively correct structure before shifting to an incorrect one. His drawings, which illustrated these shifts, helped us to visualize the connections between his analytical work and the physics context. We summarized his reasoning by focusing on his spoken language, hand gestures, and drawings.

Results

Analyzing Matthew's Moment-to-Moment Reasoning in the Water Pressure Task

In this section, we discuss Matthew's work on the Water Pressure task (adapted from Sealey, 2014) from Interview 2 (see Fig. 1). Notably, this task was novel to Matthew, minimizing the likelihood that he relied on recalled formulas or procedures from his physics or calculus courses. We first present the data Matthew produced and then analyze it using the methodological approach described earlier.

Pressure, P, applied across a surface area, A, creates a total force, F. Consider a vertical side wall of a tank with a width of 4 meters and a maximum depth of 3 feet (see picture below). Assume that the tank is full of water.



The pressure on the tank wall increases with the depth of the water according to the following law: $P = 15x$, where x is the depth of the water (the deeper the water, the greater the pressure). Given this information, describe a method for approximating the total force from the water pressure on the wall.

Figure 1. A Water Pressure task statement, Interview 2.

Let us remark that a normatively correct integral construction in this task would have the following representation $\int_0^3 15x \cdot 4\, dx$.

Matthew constructed an initial structure of the "product layer". Matthew began by writing down the given data which led him to think that "Each P is like the pressure on like a little bit of the area." He claimed that he did not know the actual relation between the quantities in the given problem situation: "I don't know the actual relation, so I'm just guessing." Then he stated: "I would say force is the integral of the pressure on a little bit of area (Fig. 2a). And umm the area is, well the area is constant essentially makes sense (Fig. 2b)."

$f = \int P dA$	dA $A = 4m \cdot 3ft$
(a)	*(b)*

Figure 2. Matthew's initial thoughts to approach the task.

Matthew stated that he did not see why the equation $P = 15x$ is relevant because "you would just plug, you would just maximize the pressure by putting in the maximum depth of the tank in for x." Hence, he obtained: "So, I guess P would just be 15 times 3, which would be 45."

Analysis. Initially, Matthew applied a set of knowledge resources to reason through the problem. He first considered pressure as a set of values acting on small areas, suggesting he saw pressure in relation to these areas, though it wasn't clear how he combined the pressure values. He then expressed uncertainty about the relationship between force, pressure, and area, and, based on a guess, produced an integral expression for force (Fig. 2a). This expression was tied to his initial reasoning about pressure acting on a small area, potentially aligning with the PdA notation. Finally, by examining the given data, Matthew concluded that (a) the area was constant (Fig. 2b), and (b) the depth variable x was irrelevant to the pressure formula, as knowing the maximum depth allowed him to maximize the pressure. This progression of knowledge resources showed how Matthew initially generated normatively correct reasoning (Fig. 2a) but later introduced additional reasoning (a) and (b) that competed with the correct interpretation.

Matthew shifted his reasoning about the "product layer" structure. Matthew continued hesitating about the relationship between the pressure, area, and force quantities. He claimed: "I do not know if it's going to be area dP, like a little bit of pressure over the entire area, or it's this total pressure over the entire space of the area." Assuming that the area quantity is constant, and the pressure quantity changes, Matthew produced another integral expression to estimate the total force (see line 2 from the top in Fig. 3):

Matthew: I think I would actually not go with this [integral expression in Fig. 2a] and do, maybe, dP times A. And then you get, because then you would pull out the area, which would be 4 meters times 3 feet and dP would be 15 dx, I guess, from 0 to 3. And I guess that would be my estimation of the total force.

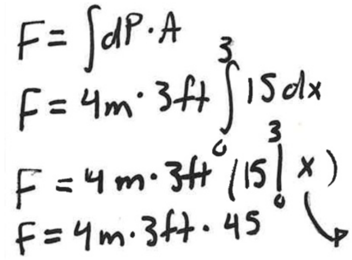


Figure 3. Matthew shifts his reasoning about the "product layer" structure.

Additionally, Matthew calculated the integral value in the force expression, arrived at $F = 4m \cdot 3ft \cdot 45$, and stated: "I am not super confident in this answer because I'd say the main problem is that I'm not understanding the relationship between force and pressure and area."

The interviewer asked Matthew if he had any other thoughts or ideas to approach the given problem. Matthew answered: "I'd say that's my final one. The reason I said F equals dP times A instead of $P\ dA$, which is area, is because the area isn't changing, like the size of the wall isn't changing, but the pressure is with the depth of the water."

The interviewer next asked Matthew about the units attached to the 45 value in his force expression. Initially, Matthew stated he did not know the units for pressure but then provided $P = \frac{kg}{s^2 ft}$, which he justified by coordinating units for area and force.

Matthew explained his shift in reasoning using drawings. The interviewer asked Matthew whether he could explain his shift in reasoning using a picture. Matthew responded:

Matthew: So, if this is like your dA (Fig. 4a), you would have the pressure value acting on that little piece of area. And then when you sum up all the little dAs, you get this (Fig. 4b). But the reason I chose not to do that is because you already know the area in terms of constants. So, I'm assuming that what it really is, is that you have like your whole area, and you just have these little bits of pressure, these dPs acting across it. And when you sum up all these dPs, you would get the pressure (Fig. 4c).

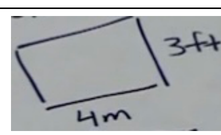
		
(a)	*(b)*	*(c)*

Figure 4. Matthew's drawings for his reasoning about the definite integral.

Following his reasoning, Matthew also produced a picture for little dAs by drawing little squares on the wall (Fig. 5a) and claimed: "I wouldn't use dA here because you already know the dimensions, it's not like something unknown."

Correspondingly, the interviewer asked Matthew to produce a picture for a term dP on the wall. Matthew drew green arrows pointing to different locations on the wall and stated that each arrow represented a bit of pressure (Fig. 5b). In few seconds, he stated that it was not the best way to represent little bits of pressure. Then he produced a picture (Fig. 5c) and added: “It [pressure] is a function of depth. I would say that you would kind of have like this different sort of like sections, I guess you could say, of the area. And like each of these sections has a dP. So then when you sum up the pressure as like a function of the depth, you would get the total.”

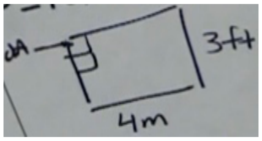
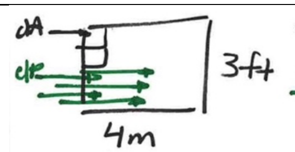
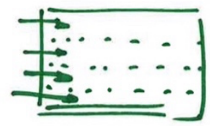
		
(a)	*(b)*	*(c)*

Figure 5. Matthew’s graphical representations of dA *and* dP *terms.*

Following Matthew’s responses, the interviewer highlighted differences in his drawings of the dP pieces. Matthew responded:

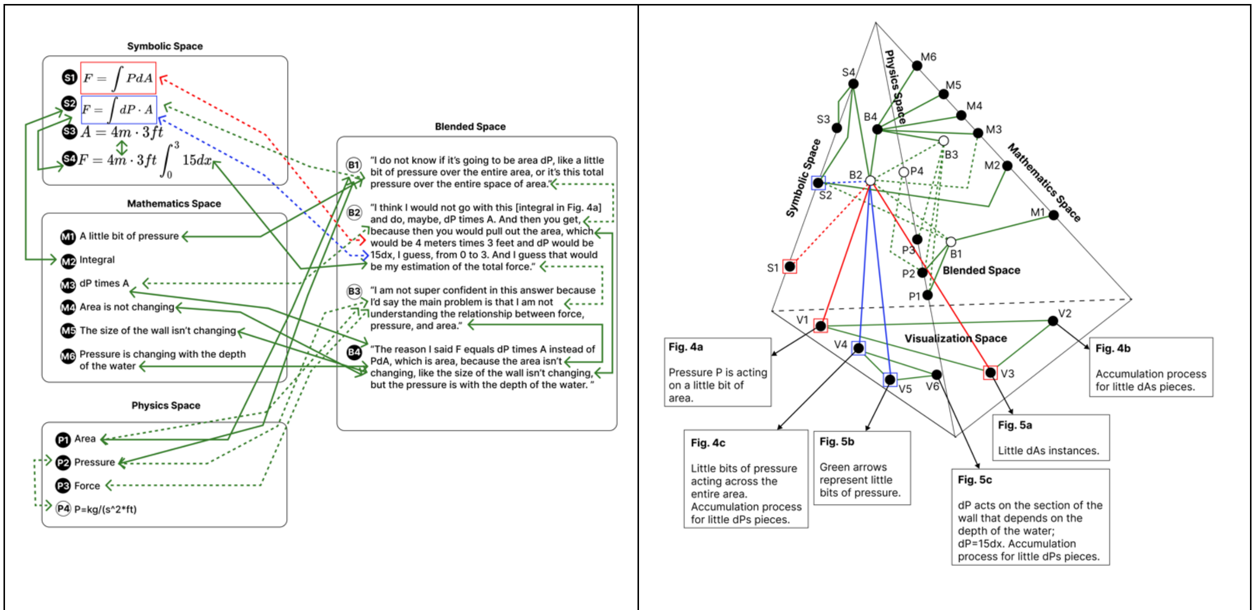
Matthew: So, originally, I just think about [pressure] in terms of the fact that it’s defined by a function of the depth. But then when I thought about that, it makes more sense to draw it like this [strips, Fig. 5c] because each bit of pressure is dependent on some sort of depth amount, like value. So, like dP is equal to $15dx$, so each little bit of depth contributes to that little bit of pressure.

After this discussion Matthew confirmed that his final graphical representation for a dA term was given by picture (Fig. 5a) and for a dP term by picture (Fig. 5c). Also, he stated that the integral expression $F = \int dP \cdot A$ made more sense to him.

Analysis. We constructed a MDCBD (Fig. 6a) to better capture a shift in Matthew’s moment-to-moment reasoning about the “product layer”. The diagram organizes selected knowledge resources and their connections related to the shift in the order they appeared in the transcript. This diagram is a subset of the expanded VAP-model diagram (Fig. 6b). The expanded VAP-model diagram includes drawings that illustrate Matthew’s shift and will be discussed further.

The MDCBD diagram illustrates Matthew’s initial uncertainty (**B1**) about his integral model in Fig. 3a. This led him to reconsider how to calculate force (**B2**), prompted by doubts about the relationship between force, area, and pressure (**B3**). By integrating knowledge resources — specifically, the constancy of the wall's area and the varying pressure — Matthew revised his integral expression to $\int dP \cdot A$ for calculating force (**S2**). His reasoning shift centered on the observation that area was “constant” while pressure changed with water depth (**B4**). Matthew also drew on knowledge of antiderivatives and definite integrals. Notably, his reasoning evolved through three models of calculating force: (a) viewing pressure as values applied to small areas, (b) considering pressure as a single value, and (c) applying small amounts of pressure to the entire area of $4m \cdot 3ft$. Finally, Matthew employed a units’ coordination approach to determine the correct units for pressure, addressing an initial uncertainty about them.

The expanded VAP-model diagram illustrates a shift in Matthew’s reasoning about the “product layer” structure and its connection to drawings. It clearly identifies the moment of transition (**S1**→**S2**) and the blended element (**B2**) that prompted the shift. During this process, Matthew produced drawings (**V4**, **V5**) that clarified to us his understanding of the expression $dP \cdot A$ and provided evidence for rejecting his initial construction, $P \cdot dA$ (“the reason I chose not to do that is because you already know the area in terms of constants”) and the corresponded drawing (**V1**, **V3**). The model also reveals Matthew’s transition from visualizing pressure distributed across the entire wall (**V5**) to focusing on the pressure acting on a horizontal section (**V6**). These drawings (**V4-V6**) ultimately led Matthew to conclude that $dP \cdot A$ was the more appropriate structure.



(a)	*(b)*

Figure 6. The MDCBD and the expanded VAP-model diagram of Matthew's reasoning about the "product layer."

The MDCBD and the expanded VAP-model diagram in Figure 6 illustrate the complexity of Matthew's reasoning and visually track the evolution of his thinking about the "product layer" structure, linking elements from five mental spaces.

Discussion and Conclusion

This study builds on the work of Oehrtman and Simmons (2023) by showing that students' reasoning about the "product layer" in definite integrals is highly complex. Our findings reveal that students integrate mathematical and physics knowledge, supported by visual representations, when reasoning about this concept.

We observed that a student, Matthew, generated two different local models from the same basic model. Initially, he constructed a normatively correct local model $P \cdot dA$, which he then extended to the global model $\int P \cdot dA$. However, he later abandoned this and developed a new local model $dP \cdot A$ with the corresponding global model $\int dP \cdot A$. This shift was guided by drawings and the physical and geometric properties described in the problem. Notably, the $dP \cdot A$ model would have been valid if Matthew had considered the pressure acting on the tank floor.

By analyzing Matthew's transitions between models and his use of drawings, we gained deeper insights into his reasoning process. While based on a single case study, these findings suggest that research on integration should pay closer attention to the diverse knowledge resources, visual representations, and their interactions that students use in reasoning about multiplication and definite integrals in applied contexts.

Acknowledgments

This research study was supported by the Graduate Student Research Competition Funding at Tufts University, Graduate School of Arts and Sciences.

References

Aspinwall, L., & Shaw, K. L. (2002). Representations in calculus: Two constructing cases. *Mathematics Teacher, 95*(6), 434–439.

Bernard, H. (1994). *Research methods in anthropology* (2nd ed.). Thousand Oaks, CA: Sage.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior, 22*, 479–495.

Bing, T. J., & Redish, E. F. (2007). The cognitive blending of mathematics and physics knowledge, *AIP Conf. Proc., 883,* 26–29.

Bueno O. (2016). Visual reasoning in science and mathematics. *Studies in Applied Philosophy, Epistemology and Rational Ethics*, 3–19.

Cui, L., Rebello, N. S., Fletcher, P. R., & Bennett, A. G. (2006). Transfer of learning from college calculus to physics courses. *Proceedings of the NARST 2006 Annual Meeting, San Francisco, CA, United States.*

Cunha, J. M., Martins, P., & Machado, P. (2020). Let's figure this out: A roadmap for visual conceptual blending. *International Conference on Innovative Computing and Cloud Computing*.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction, 10*(2/3), 105–225.

diSessa, A. A. (2006). A history of conceptual change research: Treads and fault lines. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 265–282). New York, NY: Cambridge University.

Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior, 48*, 152–167.

Fauconnier, G., & Turner, M. (1998). Conceptual integration networks. *Cognitive Science, 22*(2), 133–187.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. Basic Books.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice.* Cambridge University Press.

Hu, D., & Rebello, N. S. (2013). Using conceptual blending to describe how students use mathematical integrals in physics. *Phys. Rev. ST Phys. Educ. Res., 9*(2), 020118.

Izsák, A. (2004). Teaching and learning two-digit multiplication: coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking and Learning, 6*(1), 37–79.

Izsák, A. (2005). "You have to count the squares": Applying knowledge in pieces to learning rectangular area. *Journal of Learning Sciences, 14*(3), 361–403.

Izsák, A., & Sus, O. (2024). Knowledge resources for multiplication and challenges reasoning about the product layer. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 26th Annual Conference on Research in Undergraduate Mathematics Education*, (pp. 72-80), Omaha, NE.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122–141.

Jones, S. R. (2015a). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior, 38,* 9–28.

Jones, S. R. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann-sum based conceptions in students' explanations of definite integrals. *International Journal of Mathematics Education in Science and Technology, 46*(5), 721–736.
Jones, S. R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education, 9*(1), 8–35.
Meredith, D. C., & Marrongelle, K. A. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics*, *76*, 570–578.
Moschkovich, J. (1998). Resources for refining mathematical conceptions: Case studies in learning about linear functions. *The Journal of the Learning Sciences, 7*(2), 209–237.
Nguyen, D., & Rebello, N. S. (2011). Students' difficulties with integration in electricity. *Phys. Rev. ST Phys. Educ. Res., 7,* 010113.
Oehrtman, M., & Simmons, C. (2023). Emergent quantitative models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education, 9*, 39–64.
Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*, 230–245.
Sus, O., & Izsák, A. (2024). *New methods for analyzing students' reasoning about definite integrals in introductory physics* [Manuscript submitted for publication]. Department of Education, Tufts University.
Thompson, P., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*(1-2), 124–147.
Van den Eynde, S., Schermerhorn, B. P., Deprez, J., Goedhart, M., Thompson, J. R., & De Cock, M. (2020). Dynamic conceptual blending analysis to model student reasoning processes while integrating mathematics and physics: A case study in the context of the heat equation. *Physical Review Physics Education Research, 16*(1), 010114.
Wittmann, M. C. (2010). Using conceptual blending to describe emergent meaning in wave propagation. *Proceedings of the 2010 International Conference on the Learning Sciences, Vol. 1,* 659–666.
Yeatts, F. R., & Hundhausen, J. R. (1992). Calculus and physics: Challenges at the interface, *Am. J. Phys., 60*, 716.
Zazkis, D. (2013). Prompted and unprompted transitions between representational modes in calculus. *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.
Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D_4. *Journal for research in Mathematics Education*, 435–457.