Metacognitive Processes of Undergraduate Students During Mathematical Proof Construction

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This phenomenological case study explores the metacognitive knowledge of undergraduate students that they accessed while constructing mathematical proofs, and the related actions they take in their proof process with that knowledge. Two pairs of undergraduate mathematics majors completed two proof-construction task-based video-recorded interviews and engaged in follow-up qualitative interviews asking them to watch clips of their work and reflect on their thinking. This preliminary report discusses the qualitative data analysis process that has explored the control actions taken related to metacognitive knowledge of one pair of students during the first proof task.

Keywords: Metacognition, Problem Solving, Proof

In mathematics, conjecturing and proving new theorems is how humanity's mathematical knowledge is expanded and communicated (Ernest et al., 2016; Hamami, 2022). Undergraduate mathematics proof-based courses aim to imbue the necessary mathematical knowledge, writing and reasoning skills for students to read, construct, and analyze proofs. Development of these skills is not confined to individual courses but is an overarching goal of undergraduate mathematics programs (Committee on the Undergraduate Programs in Mathematics [CUPM], 2015). Students' proof skills impact and are impacted by both communication with others and their own reasoning about mathematics (e.g., Hanna & Barbeau, 2008). Previous studies note a relationship between one's metacognitive skills and success in problem-solving tasks (e.g., Mamona-Downs, 2002; Schoenfeld, 1994; Zhao et al., 2019). This study seeks to expand the body of mathematics education research by considering metacognitive processes of undergraduate mathematics students during the proof construction process.

I consider *metacognition* as "any knowledge or cognitive activity that takes as its object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about, one's own thinking" (Lerman, 2020, p. 608). The meta-level (where metacognitive knowledge resides) and object-level (anything external to the meta-level) interact through monitoring and control. *Monitoring* occurs when the information taken in influences the meta-level whereas *control* occurs when metacognitive knowledge impacts external cognitions and actions (Nelson & Narens, 1994).

I define *proof construction* to be the process of using a series of logical assertions deduced from a set of premises and previous assertions, and/or a set of instructions that when followed are representative of these deductions, where this collection of assertions and instructions results in an argument that "justifies a claim to oneself and others" (Blanton et al., 2003, p. 113). *Proof* is then defined to be this collection of assertions. Research has explored cognition related to the proof construction process (e.g., Hanna, 2020; Inglis & Alcock, 2012; Moore, 1994; Weber, 2001). What remains is how individuals examine their own thoughts and skills related to proof, and how their judgements impact their decisions during the proof construction.

Literature on metacognitive processes in mathematics education has predominantly focused on problem solving. It is established that metacognitive skills play an important role in students' development as problem solvers (e.g., Downs, 2002; Schoenfeld, 1992, Zhao et al., 2019).

Students employ metacognitive skills during all stages of the problem-solving process including the orienting, planning, executing, and checking phases (Carlson & Bloom, 2005).

Views differ among researchers on how problem solving and proving are related (Mamona-Downs & Downs, 2011). Some mathematics education researchers consider proving to be a subset of problem solving and use problem-solving frameworks to study proof construction (e.g., Savic, 2015; Selden & Selden, 2013; Wulan et al., 2021). However, research also suggests that some components of the proof construction process are not inherently addressed in problem solving. It remains unclear if and in what ways metacognition in problem solving compares to that in proof construction. Specifically, mathematical proofs constitute their own genre (Selden & Selden, 2013) and the language, deductive reasoning, and formatting can differ from other problem-solving tasks. These set proving apart as a specific case of problem solving, suggesting that proving requires a separate line of metacognitive research (Raman, 2003; Savic, 2015).

To gain insight into students' metacognition, I seek to identify students' metacognitive knowledge and processes during proof construction. More precisely, with this phenomenological case study, I aim to address the following research questions: What declarative, procedural, and conditional metacognitive knowledge do students demonstrate or claim to have and use during construction of a mathematical proof? And what monitoring and control processes do undergraduate mathematics students engage in during construction of a mathematical proof?

Theoretical Perspective and Methodology

I approach this research with a social constructivism theoretical perspective, concerned with individuals' constructed conceptions while also viewing their experiences and dialogue with self and others as integral to their learning process. This concern for social and cultural contexts distinguishes social constructivism from other forms of constructivism (Patton, 2002). I specifically considered intramental and intermental thinking, (i.e., the internal and external respectively) (Ernest, 2010). I claim that metacognition is a key component of intramental thinking as it centers on an individual's interactions with their own cognition.

Vygotskian social constructivism asserts the appropriation and use of signs (language) and tools (strategies) (e.g., Ernest, 2010; Vygotsky, 1987). The process of appropriating sign and tool use allows restructuring of conceptions and impacts others' knowledge through the use of tools and signs (Grossman, 1999). I claim that the process of restructuring one's own conceptions requires metacognitive monitoring, and access to metacognitive knowledge regarding what one already knows, so that alterations can be made. Additionally, the metacognitive process of control informing the individual's use of signs (language) and tools (strategies) in practice, mediate the collective appropriation of such knowledge towards agreed upon social practices.

A phenomenological study examines the "meaning, structure, and essence of the lived experience" (Patton, 2002, p. 104), in this case the phenomenon being studied is individuals' cognitive and metacognitive processes during the construction of mathematical proofs. I conducted a qualitative phenomenological case study to examine two pairs of undergraduate students' metacognitive processes as they construct proofs, through in the moment verbalization of their thinking and later reflections on their proof construction experience. The four undergraduate student participants were from proof-based undergraduate courses at a mid-size university in the United States. Prior to interviews, students were paired based on similar mathematical backgrounds and experience working on mathematics collaboratively together.

Data Collection and Analysis

As individuals' conceptions "are formed through their interactions with each other as well as by their individual processes" (Ernest, 2010), this study had pairs of participants engage in collaborative proof-construction tasks using think-aloud protocols. Each pair was given version A or version B of a task and asked to prove or disprove the claim. Version A provided a true claim, and version B provided a falsified version of the true claim. Students were audio and video recorded and remotely monitored via Zoom©, and Livescribe© pens were used to record written data. Once participants completed the task, I returned to the participants to ask follow-up questions based on my observations during the task. I then determined portions of video where metacognitive processes may have occurred by utilizing the Model of Sign Appropriation and Use (Ernest, 2010, p. 44) for possible social location and ownership of utterances. Participants were invited to an individual follow-up interview a couple days after the paired interview to watch these selected video clips, reflect on their thinking during that part of the task and answer reflection questions about their experiences.

The preliminary results presented in this paper are the analysis of Lola and Harper's experience with version B of the first task. Figure 1 is an abridged version of the task, adapted from Gilbert and Larson (2012). Note that due to a typo, I had to step in to make a minor correction to the claim that only one of k or l had to be greater than 1.

Task 1 version B prompt. A gamekeeper gives player A a positive integer. The players then take turns changing the integer they are given to a smaller integer, and then pass that smaller integer back to their opponent, player B. If the integer you received and are changing is even, the two legal moves are:

- Subtracting 1 from the integer, *or* Dividing the integer by 2

If the integer you received and are changing is odd, the two legal moves are:

- Subtracting 1 from the integer *or* Subtracting 1 from the integer, then dividing this result by 2

The game ends when the integer reaches 0, and the player who makes the last move (changing an integer to zero) wins the game.

Prove or Disprove the Claim: Assume the starting integer is given to player A. Any starting integer that when factored is written as $2^k \cdot l$, where k and l are both odd integers greater than 1, will guarantee that player A wins the game.

Figure 1. Task 1 version B

Both the task and follow-up interviews were transcribed. Data analysis and use of each code is shown in Figure 2. I first used deductive methods, keeping in mind the Model of Sign Appropriation and Use (Ernest, 2010, p. 44), to determine whether each transcript excerpt described internal thought processes (coded as Intramental), or other external happenings (coded as External). The intramental data was analyzed to determine if the subject of each excerpt was their own thinking (coded as metacognitive) or was about the object-level (coded as cognitive).

From the metacognitive data, I then sought to determine actions taken due to metacognitive reasoning, as this would be indicative of a control process. To do this, I separated the metacognitive excerpts into two categories, one for excerpts that were followed by an action (coded as having a relevant action), and one for any metacognitive knowledge stated apart from any action taken (coded as stand-alone).

From the relevant action data, inductive coding was used to categorize each indicated action. Each piece of data was individually coded by a description of the action taken, and then codes that were similar were combined. Because these actions were taken following the indication of metacognitive knowledge, they were coded as control actions.

Apart from relevant actions, I also used deductive coding on all data coded as metacognitive to determine if the metacognitive knowledge indicated was declarative, procedural, or conditional. At this point all data was placed into case files for each participant based on whose metacognitive knowledge was indicated in the transcript, whoever initiated the discussion of their reasoning.

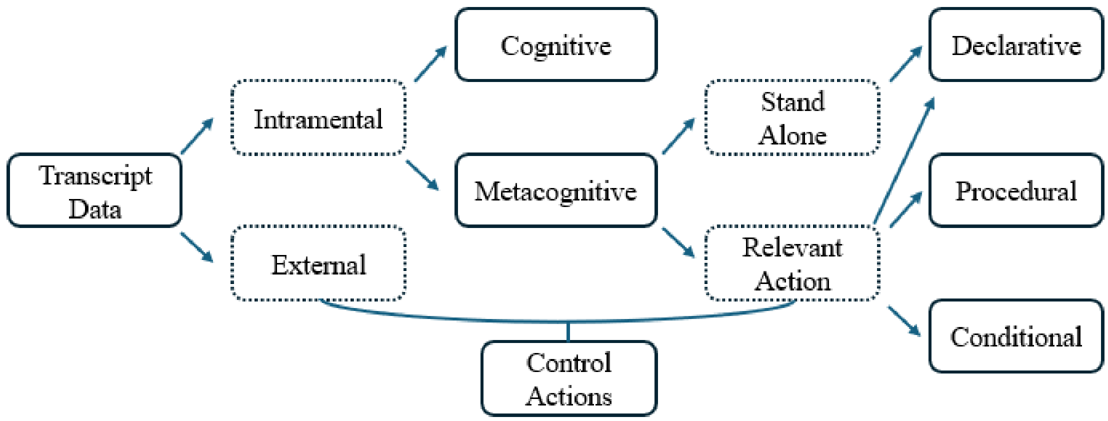

Figure 2. A Flow Chart of the Data Analysis Process

As an example of the data analysis, consider this quote from Lola, "I don't know, the most efficient way to write this? Let's - let's write out the tree and then. Yeah, so B can give two." Lola and Harper then had a lengthy discussion of how to order their disproof from the stages of their tree diagram. Since this excerpt describes Lola's internal thinking about writing, this was coded as Intramental. Because the subject of her statement was her own knowledge, uncertainty that she knew the most efficient method, this was coded as metacognitive. There was a relevant action in this statement the decision of her proof organization.

Preliminary Results

As Lola and Harper began, Harper read the task aloud. Lola then began writing generalized versions of odd numbers for k and l. Harper suggested trying some examples, with each of them as the two players in the game. Lola, in her follow-up interview, noted that she often likes to write down the given information and relevant definitions when beginning a proof. In her follow-up interview, Harper stated she often begins by trying concrete examples. In the paired interview, Lola commented trying examples might be a way to determine if a counterexample exists. They decided on the starting value six, and played the game for a couple of rounds, to work out possible outcomes from allowable moves. They determined six was a counterexample to the claim. Working through other examples, they found a starting number of eight is also a counterexample. Harper suggested drawing tree diagrams of each possible decision in the game. Lola added arrows to these diagrams to indicate which player was "giving" a number that round. Lola was the first to indicate a desire to write a proof that the claim is false using their examples. When unsure how to begin writing, Lola suggested they follow the order of how each turn played out in their tree diagram. Harper ended up being the one to write the proof on paper, but both Harper and Lola went back and forth stating what should come next in their written proof.

Throughout the task and follow-up interviews there were instances of metacognitive knowledge and related actions that were inductively coded as types of control actions. The control actions from Lola and Harper included Reading Aloud, Re-reading the Prompt, Checking Work, Re-working, Evaluating Strategy, Generalizing, Using Math Drawings, Organizing Proof, Writing Proof. A few of these codes are in Table 1 with descriptions and corresponding example quotes. Both participants engaged in most of these actions with a few exceptions. Only Harper demonstrated Reading Aloud, and only Lola demonstrated Generalizing and Writing Proof.

Table 1. A table of a few demonstrated control actions participants took.

Metacognitive Control Process Actions During Proof Construction		
Control Actions	Description	Example Transcript Data
Re-Working	Reviewing why their work was useful	"wanting to go back and run through the logic again, remembering what I was trying to prove"
Evaluate Strategy	Actively considering the value or effectiveness of their current approach	"I don't know if this general tree thing is ever going to end…Any pattern…has probably already showed up and we just didn't notice it. Let's go back to actual numbers."
Organizing Proof	Determining what will be included in the proof or the order	"I don't know, the most efficient way to write this? Let's hit let's write out the tree and then. Yeah, so B can give two…"
Writing Proof	Deciding to move from "scratch work" to writing down a proof or part of a proof	"…I write chaos all over the place on a page just to give myself a sense of what's going on…then I…write it out using more formal language and a series of steps that you can follow…"

There were notable patterns in the metacognitive knowledge utilized in each task that will be more thoroughly verified in continuing data analysis. Most often the metacognitive knowledge related to Harpers actions was procedural, related to possible strategies usefulness in this context. While metacognitive knowledge was not always clearly demonstrated, Lola frequently initiated a pause for various forms of reflection during the task. These intentional pauses may be indicative of further metacognitive monitoring and control processes that are not explicit in the data.

Discussion and Questions for the Audience

The preliminary results indicate that undergraduate mathematics students are accessing metacognitive knowledge during proof construction, and their metacognitive knowledge impacts their choices during the proof construction process. The two proof specific control actions observed indicate that there are control processes unique to proof construction. This preliminary result solidifies the need for considering metacognition in proof tasks specifically. The exploration of the declarative, procedural, and conditional metacognitive knowledge is ongoing with the second pair's data. I will complete the analysis with constant comparison of each case. My questions for the audience are: were there any instances of data in the presentation you felt should have been coded as metacognitive that were not? In your teaching and research experience, have you observed metacognitive knowledge or actions specific to proving?

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