Building Hallways to Collaborative Reasoning in Inquiry-Oriented Linear Algebra Activity

Jessica L. Smith
Utah Valley University

Matthew Mauntel
University of New Hampshire

Zachary Bettersworth
Western Kentucky University

This study examines how students engage in collaborative reasoning while working through an inquiry-oriented task on null space in linear algebra. Three students participated in a series of five online group interviews, followed by an individual interview. The task sequence involved exploring subspaces and null spaces using pathways in a school map. We analyzed interactions and identified moments where students built on or connected to one another's ideas, influenced each other's reasoning, or left space for others to contribute. Our findings suggest that students' collaborative reasoning evolved over time, with instances of adopting and leveraging peers' methods. We highlight the role of positioning in group dynamics and how students' contributions to collaborative reasoning influence their understanding of mathematical concepts. The conclusions emphasize the importance of collaborative learning environments in facilitating deeper mathematical engagement and reasoning.

Keywords: collaborative reasoning, positioning, null space, inquiry-oriented instruction

Active and collaborative mathematics classrooms have been found to benefit students' overall mathematical experiences (Freeman et al., 2014; Laursen et al., 2014; Rasmussen et al., 2020), to improve student attitudes toward math (Laursen et al., 2014; Zakaria et al., 2010), and to decrease math anxiety as compared to that related to lecture-based courses (Gholamali Lavassani et al., 2011). Active learning often involves group and whole class discussions and thus comprises students sharing their ideas and engaging with others. Indeed, collaborative math classrooms using group work have been seen as promoting equity in learning (Esmonde, 2009). However, group work can position particular students with authority in their own learning (Langer-Osuna, 2011). Within these positionings, some students can be seen as holders of mathematical knowledge and thus have their ideas explored more often and thoroughly. In contrast, others might not have their ideas explored as often. In this study, we examine the ways three students reason collaboratively in an inquiry-oriented group interview. We investigate the ways students interact with each other's ideas and the inherent positioning in those interactions. We are interested in the ways students position themselves, each other, and each other's ideas while working toward a common goal: understanding null space in the context of an experientially real task.

Conceptual Framework

Inquiry-oriented instruction is one subset of collaborative learning centering students' ways of reasoning and discourse (Kuster et al., 2017). Making students' ideas central to math learning involves navigating differences in reasoning, making these ways of reasoning public, and students interacting with and engaging in each other's ideas in small-group and whole-class discussions. Additionally, it is intended that students work collaboratively to explore, reason about, and connect these different ideas. Students work together to eventually generalize and formalize the math concept of interest.

However, by selecting which ideas to bring forward and how to sequence them, students and the instructor are implicitly assigning value to particular ways of reasoning (Langer-Osuna, 2017). In small groups, the ideas students explore and choose to bring to the class are often considered the most mathematically correct, which positions students in the group as either having mathematical knowledge or not. The ways that people are positioned can affect which ideas are elevated during problem solving (Langer-Osuna, 2016, 2018). In addition, how a teacher evaluates the merit of a student's work can affect how the group positions and assigns authority to that student. Students' positionings, whether positioned by themselves or others, are informed by factors such as social standing, academic standing, race, gender, SES, and more. For instance, a student who is seen as holding more mathematical knowledge might have their ideas privileged or leveraged more often. However, if the latter student is considered socially popular, then there is potential for their ideas to be taken up more frequently by their peers.

Collaborative learning has been defined as "the form of classroom organization in which students work together, in small groups, on a shared activity and with a common goal" (p. 57, Sam & Tan, 2011). For this study, we focus our attention on collaborative activities such as students' engagement with others' ideas, the connections students make between their ideas, and whether certain ideas were leveraged. This is because we are interested in the ways that students privilege certain ideas over others when engaging in collaborative activity. This inherently involves positioning because if there is collaboration, then students will always be saying something about their own or others' ways of reasoning, whether implicitly or explicitly. These activities have not been explicitly explored as part of collaborative reasoning. Thus, we examine students' positioning within the context of an IOLA task sequence. In our analysis, we seek to answer one research question: *How do students reason collaboratively about null space within an inquiry-oriented task sequence?*

Methods

Three students from across the Eastern United States were interviewed over the course of three weeks by the second author during five collaborative online Zoom interviews. All the students were white males who had recently completed a course in differential equations. Two of the students, Henry and Tyler, had taken linear algebra previously, and Spade had not. Each of the interviews was recorded and transcribed. PDF scans of students' written work were collected via email after each interview.

During the first four interviews, the students participated in a task sequence on subspaces designed for the IOLA curriculum (Andrews-Larson et al., in prep; Plaxco et al., 2021) that supported a conception of vectors as pathways in a school map. The end goal of the sequence was to have students investigate the null space of a linear transformation. In the task sequence, the linear transformation is conceptually represented as a matrix that transforms camera data vectors into room population change vectors. The null space of this linear transformation can be conceived as all the camera data vectors that result in no change in the population of all four rooms, which can be thought of as loops in the school map. The interview on the fifth day was designed to be a playful task where the students were asked to take the original map, rearrange the rooms and hallways, and determine the effect on the null space.

The interviews were transcribed using Zoom. For our preliminary analysis, each author reviewed each video and identified episodes of collaborative activity. These episodes were then reviewed by the group, and collaborative reasoning actions within each episode were identified.

Findings

In this section, we illustrate instances of the three ways that students collaboratively reasoned (and bold those instances) about null space while interacting with each other throughout the interviews. We begin by providing examples from the first day because all three students were still developing a working relationship with each other and the interviewer. In later interviews, all three students became more familiar with each other and their mathematical ideas. However, we still share findings from the first teaching interview specifically because our students were exhibiting collaborative activities despite the fact they were still getting comfortable with the interview format and each other. Then we provide additional examples from the later days choosing examples that best illustrate the ways our three students reasoned collaboratively.

Day One: Representing All Pathways Between Rooms

In this section, we will discuss the three main collaborative activities in which we saw students engage: one student's reasoning influencing another's, one student connecting their way of reasoning to another's, and students leaving space for another to talk. The students were asked to find a way to represent all possible solutions to movement around the West Wing. The interviewer allowed students time to think on their own and then to talk to each other. For context, Tyler was not present for about 30 minutes of the interview due to a bad connection.

Henry began by saying it reminded him of "a solution set to a matrix or something, so I'm thinking we need to have two linearly dependent vectors and then add two weights to them." Beyond connecting his reasoning to more formal linear algebra concepts, Henry was also able to situate these ideas in the context of the task. He explained that to represent movement from room C to room C, they could do the square loop and the triangular loop and that some combination of these loops would always get them from C to C. However, for A to C, they needed to add on a vector showing movement in cameras 1 and 2 in addition to any amount of loops.

Spade starts his explanation by saying he was "a little lost" but still explained how he was reasoning about the problem. He mentioned doing some sort of n-1 approach but voiced not knowing how to explain it. Henry asked if he meant representing "it as a summation or something." Spade explained that the ways you go from C to C are all the way around, so using the vector <1 1 1 0 1> as many times as you go around that path. Henry started, "Yeah, I think that's actually, I don't want to interrupt you but–actually, just keep going," **making space** for Spade to continue his explanation and contribute to the pair's collaborative reasoning. Spade explained that you just keep adding the vector as you do the path more times, again mentioning n-1 without elaboration. Henry then **made a connection between their ways of reasoning**, explaining that they both used vectors to represent the loop paths and that you can do those as many times as you want. Doing these loops continuously led to needing scalars. This interaction ends with Spade agreeing with Henry. Spade ultimately adopted Henry's way of reasoning about the hallway movement, using vector expressions to represent loops. When Tyler joined the interview again, Spade explained that each vector in the expression representing solutions to movement from C to C represents a loop one could take around the hallways. Spade added that you can do those loops multiple times, introducing the need for scalars. His explanation showed that **Henry's reasoning influenced his own** regarding pathways between rooms.

Day Two: Connecting Camera Data Vectors to Population Change Vectors

During the Day 2 interview, the students began associating specific camera data vectors (e.g., <5 3 5 4 7>) with the corresponding population change vector. For reference, the population change vector associated with the <5 3 5 4 7> camera data vector should be <6 2 -6 -2>, which means 6 students enter classroom A, 2 students enter B, etc. Spade offered his population change vector first, <6 2 -2 -2>, but qualified his response with, "I think I'm a little lost in my work." Tyler said he "almost got that but had different ones for C and D" Tyler then explained his approach to determine why his vector was different from Spade's. Tyler's approach involved a system of equations that emerged from thinking about the “inputs and outputs” for each classroom, and he explained that this also helped in determining the population change of each classroom, shown in Figure 1. Henry commented that the sum of all the entries of any population change vector should be zero, even though Tyler initially mentioned (before explaining his approach) that he knew his solution could not work because he was “adding students to the system.” When moving to the next problem, Henry wanted to represent Tyler’s system of equations as a matrix **demonstrating how Tyler’s reasoning influenced Henry’s**. Spade said he was “trying to just wrap my head around it…I understand what Tyler did, that’s what I was trying to do pretty much,” demonstrating that **Spade was also influenced** by Tyler’s input/output approach. It is worth noting that even though Spade said he was still trying to make sense of the connection between the camera data vectors and population change vectors, he was the first to volunteer a solution to both problems.

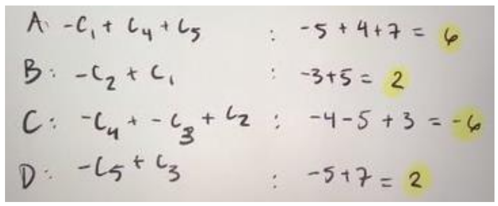


Figure 1: Tyler’s system of equations approach to determine the population change vector corresponding to the given camera data vector.

Before moving on to the second camera data vector, **Henry explained a connection between Tyler’s system of equations** and the corresponding matrix representation, with which Tyler agreed. It appeared that both Henry and Tyler were interested in using matrices to determine the population change vector for the second camera data vector. For reference, the population change vector associated with the camera data vector <5 5 3 2 3> is <0 0 0 0>. At the end of their working time, Tyler was the first to respond saying that he “did it Henry’s way,” which implied that **Tyler was influenced by Henry’s matrix reasoning**. Again, Tyler did not initially use matrices but used matrix multiplication for the second camera vector (Figure 2a). Spade then explained “I did it your way Tyler…Yup, I got zero for all of them,” demonstrating again that Spade was **influenced by Tyler’s reasoning** (Figure 2c). When asked who wanted to volunteer to share after Spade finished his explanation, Tyler responded, “Either way, we’ll probably say the same thing,” **leaving space for Henry** to explain his idea. Henry proceeded to explain his matrix multiplication approach, **making connections** to Tyler’s use of systems of equations (see Figure 2(b)). It should be noted that Tyler recognized that he should have placed the coefficient matrix to the left of the camera data vector (Figure 2a) and proceeded to give a thorough explanation of the relationship between the matrix equation and his initial systems of equations approach, demonstrating **a connection to Henry’s approach** and his initial approach.

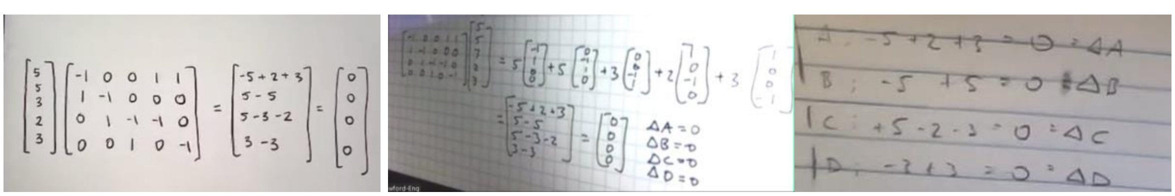


Figure 2: a. Tyler’s approach using matrices based on Henry’s explanation, b. Henry’s matrix multiplication approach that corresponds to Tyler’s system, c. Spade’s system of equations approach based on Tyler’s explanation.

Day Four: Developing Maps from a Matrix

The students were asked to construct maps corresponding to the matrix (Figure 3).

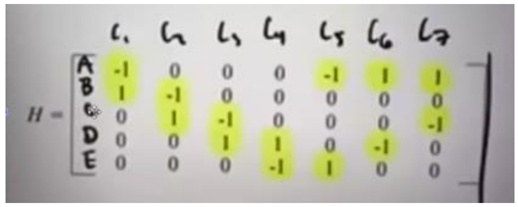
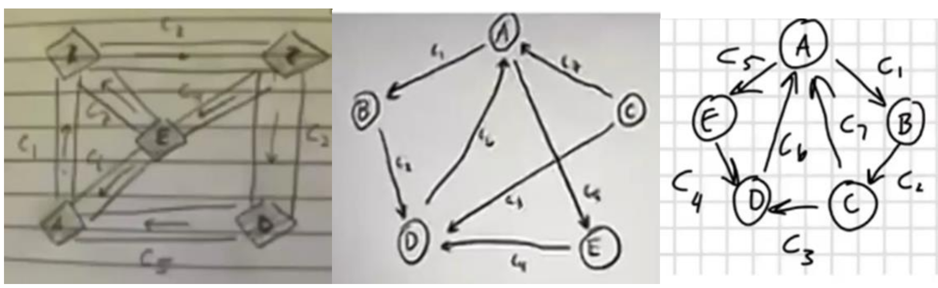


Figure 3: Matrix with Tyler's highlighting.

Each of the students worked independently and then were asked to share their work. Spade began by sharing that there was an additional room and hallway when compared to the previous map. Thus, he took the previous map, added a room in the center, and connected the hallways (Figure 4a), saying that he just "kinda did it." Tyler created a map (Figure 4b) by considering the entries in each column, that "the negative one" was people "leaving somewhere" and "the positive one" was people "arriving somewhere." This idea gave him a way to connect rooms in his map. Henry's map (Figure 4c) was similar, and he **explicitly connected his reasoning** to that of Tyler except Tyler's "went counterclockwise." After listening to Tyler and Henry explain their map, Spade redid his map, **building his own reasoning off of Tyler's reasoning**. When the students were sharing out, Henry went last, making space for the other students to share first.



4a. Spade's original map *4b. Tyler's original map* *4c. Henry's original map*

Figure 4: Student-generated hallway maps for the given matrix.

Conclusion and Discussion

In the first day of the interviews, students began to reason collaboratively by engaging in and connecting each other's ideas, even as far as to adopt another's way of reasoning. Henry was eager to make connections between his and Spade's ways of reasoning but instead made space for Spade to continue his explanation. These key ways of collaboratively reasoning continued throughout the interviews. For instance, Tyler tried Henry's method of developing a matrix equation, while Spade adopted Tyler's method of setting up a system of equations. We see from these examples that all three students reason interdependently, where students were connecting, building off, making space for, and sometimes adopting each other's ways of reasoning. These activities played a pivotal role in how students reasoned collaboratively and developed their understanding of null space. In the next stage of our analysis, we plan to examine our data more thoroughly to find additional episodes of collaborative reasoning. We are interested in further exploring the implications of students making space, or not, for each other, in addition to our other key ways of collaborative reasoning.

References

Andrews-Larson, C., Mauntel, M., Watford, M., Plaxco, D., Smith, J., & Kim, M. (in prep). *When guided reinvention and student collaboration both yield null spaces*.

Esmonde, I. (2009). Ideas and Identities: Supporting Equity in Cooperative Mathematics Learning. *Review of Educational Research, 79*(2), 1008–1043. https://doi.org/10.3102/0034654309332562

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the national academy of sciences*, 111(23), 8410-8415. https://doi.org/10.1073/pnas.1319030111

Gholamali Lavassani, M., Hejazi, E., & Khandan, F. (2011). Evaluate the effectiveness of collaborative learning on math anxiety and help seeking behaviors. *Journal of Psychology*, 15(4), 397-411.

Langer-Osuna, J. (2011). How Brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. Canadian Journal of Science, Mathematics and Technology Education, 11(3), 207–225. doi:10.1080/14926156.2011.595881

Langer-Osuna, J. M. (2016). The Social Construction of Authority Among Peers and Its Implications for Collaborative Mathematics Problem Solving. *Mathematical Thinking and Learning*, 18(2), 107–124. https://doi.org/10.1080/10986065.2016.1148529

Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education, 48*(3), 237-247.

Langer-Osuna, J.M. (2018) Exploring the central role of student authority relations in collaborative mathematics. ZDM Mathematics Education 50, 1077–1087. https://doi.org/10.1007/s11858-018-0965-x

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education, 45*(4), 406-418. https://doi.org/10.5951/jresematheduc.45.4.0406

Plaxco, D., Andrews-Larson, C. J., Smith, J., Kim, M., Mauntel, M., & Watford, M. (2021). Introducing an RME-based task sequence to support the guided reinvention of vector spaces. In *23rd Annual RUME Conference on Research in Undergraduate Mathematics Education*. MAA.

Rasmussen, C., Apkarian, N., Tabach, M., & Dreyfus, T. (2020). Ways in which engaging with someone else's reasoning is productive. *The Journal of Mathematical Behavior*, 58, 100742.https://doi.org/10.1016/j.jmathb.2019.100742

Sam, H. K., & Shim, G. T. G. (2011). The Effectiveness of Collaborative Learning in The Teaching of Form Four Mathematical Reasoning. *Sains Humanika*, 55(1). https://doi.org/10.11113/sh.v55n1.237

Zakaria, E., Chin, L. C., & Daud, M. Y. (2010). The effects of cooperative learning on students' mathematics achievement and attitude towards mathematics. *Journal of Social Science*, 6, 272-275. http://dx.doi.org/10.4236/ce.2013.42014