The Guided Reinvention of Equivalence Classes and Equivalence Relations

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We present a preliminary local instructional theory of students' guided reinvention of equivalence classes and equivalence relations that has resulted from engaging in one iteration of the design research cycle. We leverage the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991) to design a task sequence in an experientially real setting that can guide students to reinvent equivalence classes, equivalence relations, and their properties. In this paper, we present our revised task design describing the rationale, task implementation in a classroom setting, and analyses of student reasoning. Further, we conclude by discussing our plan to reimplement the tasks in another iteration of the design research cycle.

Keywords: Realistic Mathematics Education, Equivalence, Guided Reinvention

Equivalence is ubiquitous in mathematics, particularly in abstract algebra. Students are introduced to notions of equivalence, like equality, as early as elementary school (e.g., Alibali et al., 2007; Kieran, 1981; Knuth et al., 2006). As they progress through their mathematical careers, students encounter more complex notions of equivalence like congruence and isomorphisms in abstract algebra. Researchers have explored undergraduate students' interpretations of notions of equivalence (e.g., Cook et al., 2021; Mirin, 2019) and reasoning about equivalence across domains (Cook et al., 2022). More domain-specific research has investigated productive ways of thinking about equivalence in combinatorics (Lockwood & Reed, 2020) and the notion of sameness in the context of isomorphism and homomorphism in abstract algebra (e.g., Rupnow, 2019). Despite this recent work, little is known about how students reason about equivalence relations and equivalence classes in abstract algebra. We aim to contribute to this literature through our design research-informed (Gravemeijer et al., 2003) local instructional theory of students' guided reinvention of equivalence relations and equivalence classes. Guided reinvention (Freudenthal, 1991) is an instructional design principle from Realistic Mathematics Education involving "an instance of psychological explication whenever students already possess some intuitive or less formal understanding from which they construct formalized meanings" (Dawkins, 2015, p. 67). Scholars have investigated students' reinvention of various abstract algebraic concepts, like groups (Larsen, 2009, 2013), quotient groups (Larsen & Lockwood, 2013), isomorphisms (Larsen, 2009, 2013), rings (Cook, 2012), integral domains (Cook, 2012), fields (Cook, 2012), (ir)reducible elements (Serbin et al., 2024), and unique factorization domains (Bae et al., 2024). However, scholars have not yet explored students' guided reinvention of equivalence relations and equivalence classes, which we address. In this preliminary report, we present our revised task sequence and analysis of student reasoning after one iteration of the design cycle. We address these research questions: *Which ways of reasoning do students engage in as they reinvent equivalence relations and equivalence classes? How does the task sequence in this local instructional theory evoke these ways of reasoning?*

Theoretical Perspective

Design research (Gravemeijer et al., 2003) involves a cycle by which researchers design an instructional task sequence, implement it in an instructional setting with a teacher-researcher guiding students, analyze the student reasoning that results from engagement with the task sequence, and revise the sequence to improve its subsequent use. Design research can be used to create a *local instructional theory* (LIT), "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2012, p. 107). Realistic Mathematics Education (RME) is an instructional design theory that can inform such design research (Freudenthal, 1991; Gravemeijer, 1999). RME has three design heuristics that researchers consider as they design task sequences: experientially real context problems, guided reinvention, and emergent modeling. RME-informed task sequences have problems set in experientially real contexts that have accessible entry points for students to engage in intuitive mathematical activity. These tasks should also provide the opportunity for students to create a model (e.g., a physical object, manipulative, or symbol) of their mathematical activity that they can use as a model for further abstract mathematical reasoning. This shift in the emergent model, accompanied by progressive mathematization, supports the student in reinventing a mathematical concept. Students "reinvent" a mathematical concept with the guidance of a teacher by progressively formalizing their informal, intuitive understandings. Overall, this design theory of RME informed the design of our LIT.

Methods

Our design research (Gravemeijer et al., 2003) began with designing the RME-informed task sequence to engage students in the guided reinvention of equivalence relations and equivalence classes. We aimed for them to develop intuitive, informal understandings of those concepts and transition to formally define those concepts. We conducted a teaching experiment (Cobb, 2012), where we implemented the task sequence in a class setting, analyzed students' reasoning about the tasks that were evident in recordings of class discussions, and revised the task sequence and its rationale to potentially better support students in reaching the intended learning outcome.

We collected data in a Mathematics for Secondary Teachers course at a large public university in the southern US during the Spring 2024 semester. This course contained 23 students who were junior and senior mathematics majors. The task sequence was administered in the beginning of the hybrid course over four classes; two in-person and two on Zoom. These classes were recorded and transcribed. The recordings could not capture group work discussions, so we only analyzed the student reasoning shared in whole-class discussions. The projected/shared screen was recorded. Pictures of student work on whiteboards were also collected. We performed a qualitative video analysis (Lesh & Lehrer, 2012), in which we wrote content logs of the class discussion videos and memos of students' ways of reasoning evident in the videos. These memos also included reflections of how the tasks seemed to evoke these ways of reasoning. Each author individually wrote these content logs for the four classes, and we met to compare our interpretations of the videos and our analyses of student reasoning. We discussed how effectively the tasks guided the students' reinvention and decided how to revise them.

Results

We present our revised task sequence based on this analysis. We describe each task, its rationale, and examples of student reasoning exhibited on the task. We then present additional tasks 6-11, which were added to the task sequence during the revision, along with their rationale.

Task 1: Coloring task

Our LIT begins with a number line coloring task, which leads students to create a set of equivalence classes under congruence modulo 3 on the set of integers. The task states that each number on the number line is colored either red, blue, or green and outlines two rules: 1) The negative (opposite) of a red number must be colored blue and 2) the sum of two blue numbers (not necessarily distinct) must be colored red. Students are tasked to find a coloring of the number line that meets these conditions. A coloring using all three colors and meets these conditions will successfully divide the number line into a repeating green, red, blue (or green, blue, red) pattern (see Figure 1). This task engages students in an experientially real task setting of coloring an integer number line. We aimed for students to use the colored number line as an emergent model of their activity that they could mathematize to reinvent equivalence classes.

In line with the exploratory nature of the task, students used a trial-and-error method, leading to correct and incorrect strategies. A common incorrect strategy was to start by making all negative numbers blue and all positive numbers red (or vice versa). When using this strategy, students noticed that it led to problems and contradictions like: "We chose red to be the positive numbers and then the blue ones to be negatives, and when we add two negative numbers, we can't end up getting positive." The group noticed that since they colored the negative numbers blue, their sum should be colored red (condition #2) which means the sum should be a positive number. However, given that the sum of two negative numbers will always be negative, this leads to a contradiction. A successful strategy that was used by a couple of groups was to build up the coloring based on the two conditions, as one student explained it: "We started with 1 and from there we went and said 1 plus 1, since the sum of it is 2, then it's red [condition 1]. So since 2 is red, then the opposite has to be blue [condition 2], so -2 would be blue." This system resulted in some numbers not being assigned blue or red, so the group decided to color these numbers green. Once students shared this strategy, the instructor had the rest of the groups use it to color their number lines to develop the class's shared understanding of the strategy.

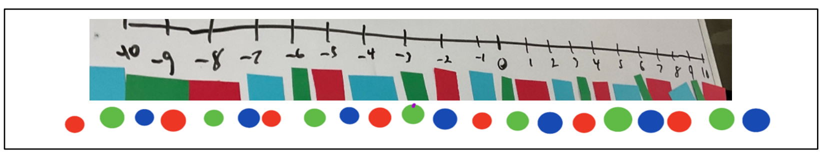

Figure 1. Students' patterned colorings (green-blue-red or green-red-blue) of the integers number line in task 1.

Task 2: Make a rule for each color

Task 2 guides students to develop rules to determine what color any integer will be according to the pattern. This task anticipates the formalization of the equivalence classes since the sets of integers with green dots, blue dots, and red dots are each equivalence classes on $\mathbb{Z}$. We saw two strategies for articulating a rule. First was a student's "jumping the number line" strategy:

> Whenever it starts at 0, you can just add 3, so it would be 0, 3, and then plus 3, 6, plus 3, 9, plus 3, and so on and so forth. However, if you start at 1, then 1 plus 3 would be 4, which is red. 4 plus 3 is 7, which is red. And then 7 plus 3 is 10. And the same thing works for 2. So, 2 plus 3 is 5, 5 plus 3 is 8, and so on.

Using the number line model, the student considered the starting point for a color as 0, 1, or 2 and made jumps of size three to find the next number with the same color. Others mathematized this idea with set notation, writing $\{3k|k\in\mathbb{Z}\}$ to represent green, $\{1+3k|k\in\mathbb{Z}\}$ for red, and $\{2+3k|k\in\mathbb{Z}\}$ for blue. The second student strategy involved modular arithmetic. Students noticed the numbers could be distinguished by their remainder when divided by 3. As their rule, one group wrote "*Let n ∈ $\mathbb{Z}$, for this to find the nth color we use, green: (n mod 3) = 0, red: (n mod 3) = 1, blue: (n mod 3) =2*". To develop a shared understanding of these rules, the task next had students use these rules to determine the color 534, 2156, and -74 (green, blue, red, respectively).

Task 3: Develop an informal definition of equivalence and equivalence class

For the next task students develop a personal definition of equivalence with the goal of 1) connecting the dot activity to the notion of equivalence and 2) begin transitioning to more formal mathematics. Most students' personal definitions of equivalence involved the idea of sameness; for example, "Classified as the same under some condition" or "They produce the same results with certain conditions." The instructor prompted students to describe what their definitions had in a common and a student suggested "sameness." The instructor then asked students to elaborate on whether elements in the same class had to be exactly the same. A student said, "It doesn't necessarily have to be the same exact thing…we have multiple different objects that you can have that will satisfy these conditions. As long as those conditions are met, you have an equivalence class." The instructor connected students' informal notions of equivalence to the number line coloring activity, noting that dots of the same color are equivalent, so integers of the same color are in the same equivalence class. The instructor then wrote an informal definition of equivalence class as *the set of elements that are equivalent to each other under some rule*.

Task 4: Establish a rule for two integers being equivalent modulo 3 and use it

After the instructor claimed students' green, blue, and red-dot equivalence classes $\{3k|k\in\mathbb{Z}\}$, $\{1+3k|k\in\mathbb{Z}\}$, $\{2+3k|k\in\mathbb{Z}\}$ were sets of elements that are equivalent modulo 3, task 4 prompts students to complete the claim, "For any $a, b \in \mathbb{Z}$, a is equivalent to b modulo 3 if and only if:". The goal of the task is for students to create the equivalence relation for equivalence modulo 3. Students contributed a variety of answers like "a and b have equal remainders [when divided by 3]" and "3 divides $(a-b)$." After students shared their ideas, they chose a rule and used it to determine whether 216 and 487 were equivalent modulo 3. To ensure students could use these rules, the task prompted students to find two integers that are not equivalent modulo 3 using their rule. Students successfully found a non-example and a correct justification. One group used 11 and 13 and explained that the remainder is not the same when dividing by 3. This task helped students recognize that equivalence classes of integers with the same color contained integers that were equivalent according to some rule, which will be formalized as an equivalence relation.

Task 5: Relating Equivalence Classes and Partitions of $\mathbb{Z}$ and Defining Partition

For Task 5, students discuss what they notice about red, green, and blue dots. The goal of this task is for students to use their number line model to develop an intuition for the properties of equivalence classes on $\mathbb{Z}$, namely that they form a partition of $\mathbb{Z}$. This activity anticipates the formal definition of partition: A partition of a set X is a collection of nonempty subsets of X such that (i) the union of those subsets is X and (ii) the subsets in the collection are pairwise disjoint. Students first noticed the pattern of the colored dots on the number line repeating green, red, blue. They noticed the cardinality of each of the sets of {red dots}, {green dots}, and {blue dots} was infinite. The instructor elaborated: "We noticed that this pattern goes on for all of the integers, and so that means if I put together all those sets, of the greens, the reds, and the blues, I get all of the integers." She prompted students to restate that idea using more precise language or notation. A student claimed, "the union of equivalence classes makes up the set of integers." Another student said: "each set is distinct from each other since one is just red, one is just blue, and one is just green." The instructor revoiced that indeed the sets are distinct and disjoint, and she pointed out that this is a special property she wanted them to notice. She mentioned that this has to do with the intersection and asked what the intersection of these sets is. Three students replied that it is the empty set, so the sets have no common elements. The instructor claimed that these two criteria were axioms in the definition of a partition and guided students to define partition. She asked why the color dot sets form a partition of $\mathbb{Z}$. Students replied it was an assumption of task 1 that a number cannot be more than one color, which implies the sets are disjoint. Also, since all integers could be assigned a color, the union of those sets is $\mathbb{Z}$.

Newly Designed Tasks

We designed new tasks to conclude the sequence: *Task 6)* Find examples and non-examples of partitions of $\mathbb{Z}, \mathbb{R}$, and $\mathbb{N}$. *Task 7)* Use the following rules on $\mathbb{Z}$ to create subsets of elements that are equivalent under the rule. Do these subsets form a partition of $\mathbb{Z}$? If yes, verify using the definition of partition. If not, state what part of the definition is not satisfied. $\mathrm{x} \sim \mathrm{y}$ iff a) $\mathrm{x}^2 = \mathrm{y}^2$, b) $\mathrm{x} = \mathrm{ky}, \mathrm{k} \in \mathbb{Z}$, c) $\mathrm{x} \neq \mathrm{y}$, d) $\mathrm{xy} \geq 0$. *Task 8)* Rules with corresponding equivalence classes that form a partition of $\mathbb{Z}$ are called equivalence relations. Part 1: What properties do these rules need to have to be equivalence relations? Part 2: Create a definition for an equivalence relation. *Task 9)* Now that we have defined equivalence relation, we can refine our definition of equivalence class to make it more formal. *Task 10*) Consider this collection of subsets for the given set. Does this collection form a partition? Could these subsets form equivalence classes? Use the definitions of equivalence relation, equivalence class, and partition to justify your claim. Ex: $\{1,2\},\{2,3\} \subset \{1,2,3\}$. *Task 11)* a) What do you notice about the relationship between equivalence relations and partitions? b) Write a conjecture that formalizes what you noticed.

These new tasks guide students to 1) reinvent and formalize the definition of equivalence relation and 2) articulate the connection between equivalence classes and partitions of a set. Task 6 gives students an opportunity to become familiar with the definition of partition that is established in Task 5. Task 7 begins students' reinvention of equivalence relation by exploring different rules that do and do not satisfy the three properties of equivalence relations (reflexive, symmetric, and transitive). In Task 8, students reflect on this work and articulate these three properties, which they then use to write a definition of equivalence relation. In Task 9 the instructor uses this definition to redefine equivalence class formally (*Let ~ be an equivalence relation on A. Let $x \in A$ The equivalence class of x is $[x] = \{y \in A \mid y \sim x\}$).* In Task 10, students explore subsets of a set and are guided to notice how subsets that fail to be partitions also fail to be equivalence classes since their corresponding relation fails properties of equivalence relations. Task 11 guides students to articulate this and establish the Fundamental Theorem of Equivalence Relations: Let A be a non-empty set. 1) *if ~ is an equivalence relation on A then the set of equivalence classes of ~ forms a partition of A.2) if $\{A_i \mid i \in I\}$ forms a partition of A then there is an equivalence relation on A whose equivalence classes are the sets A_i, $i \in I$.*

Discussion

This preliminary report shares our findings from only one iteration of the design cycle in our attempt to create a local instructional theory on the guided reinvention of equivalence relations, equivalence classes, and partitions. Our analysis of student reasoning on the tasks informed our revisions and additions to the task sequence. The added tasks serve to further engage students in defining (Zandieh & Rasmussen, 2010), which was absent from the first implementation of the tasks. To further refine the tasks, we will implement the tasks in a classroom setting again as part of a new iteration of the design cycle. We will provide a theoretical contribution to the field regarding characterizations of key ways of reasoning that are milestones in reinventing equivalence classes, equivalence relations, and partitions. We hope for instructors to be able to use our task sequence to guide students to reinvent those concepts. We pose this discussion question to the audience: What feedback do you have for our design of the task sequence?

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