Pedagogical Moves Related to Analogy that Support a Unified Understanding of Eigenequations in a Quantum Mechanics Class

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We analyze the pedagogical moves related to analogies used by a quantum mechanics instructor to support a class community in developing a shared unified understanding of eigenequations across the contexts of linear algebra, spin, energy, and position. We characterize an instructor's pedagogical moves as he engaged students in analogical reasoning. Some moves include posing tasks conducive to analogizing; preparing, soliciting, and scaffolding students' participation in analogical reasoning; using deictic gestures and inscriptions; juxtaposing symbols representing the analogized concepts; and explicitly highlighting the sameness of the analogized concepts. We exemplify these pedagogical moves with analytical descriptions of illustrative class episodes. We discuss how these moves can support the class community's expansion of their common ground by fostering the development of the class's shared unified understandings of eigenequations.

Keywords: Pedagogical Moves, Analogy, Eigentheory, Linear Algebra, Physics

Quantum mechanics students reason about eigentheory in various contexts, including spin, energy, and position, with eigenequations forms such as $\hat{x}|x_i\rangle = x_i|x_i\rangle$, $\hat{S}_z|+\rangle = \hbar/2\,|+\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$. It is nontrivial to abstract the shared structure of these eigenequations and recognize them as instantiations of the same concept with analogous interpretations. Reasoning about a concept across contexts entails having a *unified understanding* of the concept: a coherent awareness of similar structure, a recognition of distinct instantiations as examples of the concept, and an ability to reason about the concept without reference to specific contexts (Lee & Heid, 2018; Serbin, 2023). Researchers have characterized students' unified understandings in various math contexts (e.g., Bagley et al., 2015; Cook et al., 2023; Melhuish et al., 2020), but there is little research on how instructors can help students develop unified understandings in physics. Recognizing eigenequations from varied contexts as instantiations of the same concept can be facilitated by analogizing (Gentner, 1983; Glynn, 1989; Richland et al., 2004). By analogizing across spin, energy, and position, physics students can generalize the structure of eigenequations that correspond with the mathematization of those quantum mechanical systems. In this study, we investigate how an instructor's analogizing might contribute to the class's taken-as-shared (Saxe & Farid, 2021) unified understanding of eigenequations. Our research question is: What pedagogical moves related to analogies can be used by a quantum mechanics course instructor to support the class community in developing a shared unified understanding of eigenequations?

Literature Review

Reasoning about eigenequations and their objects (operator, eigenvectors, eigenvalue, equal sign) and operations (matrix-vector multiplication and scalar-vector multiplication), individually or in coordination, is conceptually complex (Thomas & Stewart, 2011; Wawro et al., 2019). For instance, Wawro et al. (2024) found that for $Ax = \lambda x$, quantum mechanics students held either a *relational view* of the equal sign (the resulting object Ax on one side of the equal sign is the same as the resulting object λx on the other side) or a *functional view* of the equal sign (the matrix A acted on or transformed an input x and produced λx). Physics students must make sense of these coordinated constituent objects in varied quantum mechanical systems (Karakok, 2019; Wawro et al., 2024) and interpret eigenequations in terms of their physical referents (e.g., Gire & Manogue, 2012; Her & Loverude, 2020). This may involve recognizing eigenequations like $\hat{S}_z|+\rangle = \hbar/2|+\rangle$, $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and $\hat{x}|x_i\rangle = x_i|x_i\rangle$ as all sharing a structure or interpreting the symbols associated with that structure as meaning "transformation that reproduces the original" (Dreyfus et al., 2017, p. 10) or "to operate is to act" (Pina et al., 2024, p. 7). Physics students sometimes perceive conceptual incompatibility between the meanings they hold for eigenequations in different contexts. For example, interpreting an eigenvector as being stretched by λ is nonsensible in a spin context where eigenstates are of unit length (Wawro et al., 2024).

There is little research on how instructors can support students in cross-contextual reasoning about eigenequations. Karakok (2019) demonstrated a physics professor's focus on supporting students' understanding of objects within an eigenequation by frequently asking students, "What kind of a beast is this?" (p. 9) while pointing to symbols in $A\vec{v} = \lambda\vec{v}$. The professor's explicit focus on symbols seemed to foster students' understandings of them. Attending to meanings of symbols in eigenequations may support students in developing unified understandings of them.

Theoretical Perspectives

Having a *unified understanding* of a concept involves recognizing seemingly disparate notions from different contexts as instantiations of the same concept with a shared overarching structure (Lee & Heid, 2018; Zandieh et al., 2017). Researchers have characterized students' unified understandings of mathematical concepts across contexts in linear algebra and abstract algebra, such as functions, linear transformations, homomorphisms, inverses, identities, and factorization (Bagley et al., 2015; Cook et al., 2023; Lee & Heid, 2018; Melhuish et al., 2020; Serbin, 2021, 2023; Zandieh et al., 2017). These studies documented how students reason about the relationships, similarities, or shared properties among instantiations of a concept (e.g., additive or multiplicative inverses) and recognize them as the same concept with the same overarching structure that holds regardless of context. Such reasoning can develop as students "unify their existing understandings of seemingly unrelated mathematical concepts by identifying structural similarities among the concepts, thereby increasing the extent to which the relations among the concepts in their minds become more coherent" (Serbin, 2023, p. 4).

Developing unified understandings of a concept may involve *analogical reasoning*, "the ability to perceive and operate on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar" (Richland et al., 2004, p. 37-38). Gentner (1983) characterized analogy as a mapping from a source domain to a target domain that preserves relations between objects and attributes in them. A *source domain* is a collection of knowledge that can be applied in a new domain, the *target*, to develop new knowledge. *Objects* are whole entities, parts of entities, or a collection of entities. *Attributes* are characteristics or properties of objects, and r*elations* are associations between objects, attributes, or other relations. When analogizing, one maps objects, attributes, and relations from the source by corresponding them with objects, attributes, and relations in the target and drawing inferences about them (Holyoak & Thagard, 1989; Novick & Holyoak, 1991; Richland et al., 2004). Hicks (2020) classified analogical reasoning activities as: exporting aspects from the source, importing aspects to the target, recalling knowledge about the source, distinguishing differences between the source and target domains, and adapting the target to accommodate for those differences. Glynn's (1989) teaching with analogies model has six instructional strategies: "(1) introduce target concept, (2) recall analog [source] concept, (3) identify similar features of concepts, (4) map similar features, (5) draw conclusions about concepts, and (6) indicate where analogy breaks down" (p. 208). Instructors can use these strategies as they engage students in analogizing.

Finally, *common ground* (Clark, 1996; Saxe et al., 2015) includes the, at best, taken-as-shared "knowledge of word meanings and norms for communication, shared knowledge that enables successful communication and coordinated action" (Saxe & Farid, 2021, p. 640). Common ground "gets built up in strata" (Clark, 1996, p. 120), where "every new piece of common ground is built on an old piece" (p. 119). It is constituted over time via communicative displays by class members, thereby expanding the class's taken-as-shared understandings.

We coordinate the theories of unified understandings, analogical reasoning, and common ground to characterize how a professor's pedagogical moves facilitate a class in developing a taken-as-shared unified understanding of eigenequations. Analogizing instantiations of a concept is a way for instructors to help expand the class's common ground and develop a taken-as-shared unified understanding of that concept. We consider an analogy's source domain to be a subset of a class community's common ground. An instructor can make analogies by mapping objects, attributes, and relations from that source to a target. In mapping these objects, attributes, and relations to the target, new objects, attributes, and/or relations are created and can become part of the common ground. The common ground thus expands, and the target becomes part of a new source from which further analogical mappings can be made. This common ground expansion by analogizing contributes to the class's development of a taken-as-shared unified understanding.

Methods

Data for this study come from transcribed videos of whole-class discussions and lectures in a senior-level quantum mechanics course at a public university in the northeastern US. The class met for 50 minutes and used a spins-first textbook (McIntyre, 2012). We analyzed 22 class days. We found segments in which the instructor analogized eigenequations and enacted pedagogical moves that could support the class in developing their understanding of eigenequations. We took such instances in the data as indicative of the class community expanding its common ground for eigenequations, such as by recognizing different forms as instantiations of the same overarching concept while attending to any shared property, use, or aspect of structure. We analyzed the episodes in terms of Gentner's (1983) structure-mapping theory for analogies. We identified the given analogy's source domain, target domain, and what objects, attributes, relations, or higher-order relations were being mapped between to the two domains. Such analyses helped us answer our research question by indicating the class community's progress toward developing a taken-as-shared unified understanding of eigenequations, which is part of the class's common ground.

We used deductive and inductive coding (Miles et al., 2013) to identify pedagogical moves in the segments. *Pedagogical moves* are "discursive acts an instructor performs during instruction to accomplish an instructional goal" (Serbin et al., under review). We focused our analysis on instances when the instructor gave vocal, written, or gestural communicative displays in which he described at least two different concepts with similar features. We made an a priori codebook containing pedagogical moves for supporting analogical activity from Hicks (2020) and Glynn (1989), from which we deductively coded. We then, through inductive coding, created new codes to describe other pedagogical moves in the data. These new pedagogical moves are given in Table 1 and are exemplified in the Results. We wrote memos (Maxwell, 2013) on how those moves seemed to support the class's development of a unified understanding of eigenequations.

Results

We summarize two class episodes from Day 5 and Day 19 that illustrate Dr. Holmes's pedagogical moves that occurred while the class engaged in analogizing eigentheory concepts.

Table 1. New Pedagogical Moves for supporting Analogical Activity

Pedagogical Move	*Description*
(1) Posing a task conducive to analogizing	Assigns a task or problem that prompts students to engage in analogizing two or more concepts.
(2) Preparing students to look for similar objects, attributes, or relations in source and target	Primes the students to identify features shared by two concepts, perhaps by suggesting there is a similarity that the students should find.
(3) Scaffolding students in identifying and mapping objects, attributes, and relations	Provides support to students as they analogize two concepts, e.g., asking guiding questions that direct students' attention to the shared features of the concepts.
(4) Juxtaposing symbols representing objects in the source and target	Writes symbols that represent concepts being analogized in a way that the symbols are aligned or near each other.
(5) Using deictic gestures	Uses their body to make gestures (McNeill, 1992) that emphasize corresponding aspects in the source & target.
(6) Using deictic inscriptions	Writes an inscription that points to or emphasizes corresponding aspects in the source and target domains.
(7) Soliciting student participation to map objects, attributes, and relations from the source to the target	Invites students to verbally contribute their ideas in the class discussion as they engage in analogizing aspects in the source and target domains.
(8) Explicitly highlighting sameness of aspects in the source and target	Verbally claims that two aspects in the source and target domains are equivalent or the same.

Analogizing the Eigenequation $Av = \lambda v$ with the Spin Eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

On Day 5, the instructor, Dr. Holmes (pseudonym), used several pedagogical moves to guide the class in analogizing the canonical eigenequation $Av = \lambda v$ and the spin eigenequations:

Dr. Holmes: When you have a matrix [points to A, see Figure 1a], you have eigenvectors [points to v on the right side], right. Do people remember what what's special about an eigenvector, for a matrix, for a given matrix? What makes it eigen-esque-ish?

Student: When it's operated on by A, it only scales.

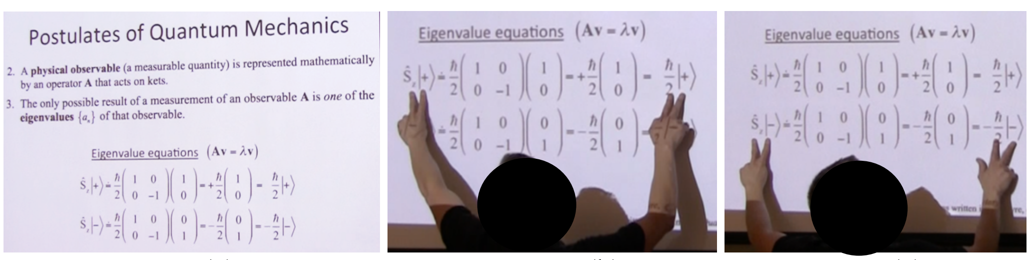
Dr. Holmes: That's right… eigenvectors don't rotate at all, they only scale. If you operate A on this particular eigenvector [points to Av], all you get is some scaling factor times the original vector [points to λv], alright. That's the magic. Those are magic vectors.

Dr. Holmes *recalled aspects of the source domain*, which included the canonical eigenequation and its operator, eigenvectors, and eigenvalue. He *solicited student participation* to *recall* these objects, relations among them (the matrix-vector relationships), and their attributes ("it only scales," "magic vectors"). He called out the ubiquity of $Av = \lambda v$, claiming, "we can write these for our operators," *explicitly highlighting the sameness* of the eigenequations' existence for operators in the source and target domains. He *introduced the target domain*, which included the spin eigenequations and their components. These pedagogical moves of *recalling, soliciting, highlighting*, and *introducing* prepare the class community to expand its common ground by setting the stage to introduce eigenequations for the first quantum mechanical context of spin.

Dr. Holmes: This equation [points to $Av = \lambda v$] is sort of canonical, right. People write these all the time for linear algebra, for matrix-vector relationships. And we can write these for our operators. I have the $S\ z$ operator … we're gonna have the value, we're gonna have the matrix, and we're gonna have the operators … You operate the $S\ z$ [points to $\hat{S}_z$ then $|+\rangle$] on the up spin state. Right, we have this matrix [points to matrix]…and you operate it on the 1, 0 state [points to $\begin{pmatrix}1\\0\end{pmatrix}$]… if you do that, you get h bar over 2 times the vector you started with. So this is the eigenvalue part [points to two ends of equation (Figure 1b)]. Same thing for this, you get a negative h bar over 2 on the bottom [points to two ends of equation (Figure 1c)] for the down state.

He *identified* and *mapped (exported and imported)* similar objects (e.g., value, matrix), relations, and attributes from the source to the target domain. He *mapped the relation* between the operator and the eigenvector by claiming, "You operate the $\hat{S}_z$ on the up spin state," which was parallel to what he said about the canonical eigenequation: "You operate A on this particular eigenvector." He also *mapped a relation* by saying, "you get h bar over two times the vector you started with" which was similar to his claim: "All you get is some scaling factor times the original vector."

Dr. Holmes also used the moves of *gesturing* and *juxtaposing symbols* while analogizing. On the slide, he *juxtaposed the placement of the symbols representing objects in the source and target*. He vertically aligned the two spin eigenequations underneath the canonical eigenequation (see Figure 1a), which had the potential to support the class in recognizing the shared structure of the different eigenequations as matrix-eigenvector product on the left and eigenvalue-eigenvector product on the right. Dr. Holmes accompanied this with *using deictic gestures.* He used two fingers on each hand to simultaneously point to each of the two objects ($\hat{S}_z$ and $|+\rangle$) on the left and the two objects ($\hbar/2$ and $|+\rangle$) on the right (see Figure 1b). He then lowered his hands to point to the corresponding objects in the other spin eigenequation (see Figure 1c). These deictic gestures helped *map the relations* between the objects in one eigenequation to the other, so the class could see that those relations were preserved. We posit these moves of using juxtaposition and deictic gestures could serve to expand the class's common ground, thus contributing to the class's development of a unified understanding of eigenequations. The class had an opportunity to recognize different eigenequations as instantiations of the same concept with the shared structure regarding the objects within the eigenequations and the relations among those objects.



(a) (b) (c)

Figure 1. (a) Slide projected during class on day 5. (b, c) Dr. Holmes's deictic gestures juxtaposing eigenequations

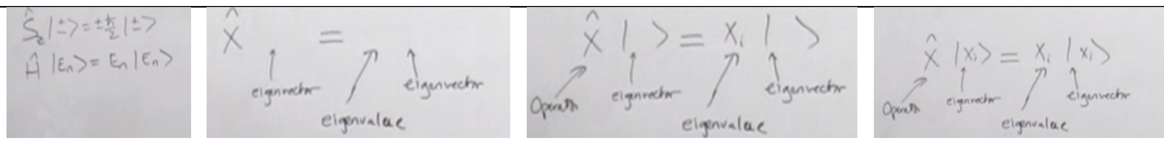
Analogizing the spin and energy eigenequations with the position eigenequation

On Day 19, Dr. Holmes used various pedagogical moves to help the class analogize the source spin and energy eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ with the target position eigenequation $\hat{x}|x_i\rangle = x_i|x_i\rangle$. He first *posed a task conducive to analogizing*: "Write down an eigenvalue equation for an operator $\hat{x}$ that represents (1-D) position." He launched the task by explaining: "We've measured all these other things… spins…and energies, right? But let's say we want to measure the position of a particle… If it's a thing we're measuring, then it has all the properties of the other things we're measuring." He *recalled the source domain* and *prepared students to look for similar attributes* of objects in the source and target domain. After students worked on the task, Dr. Holmes wrote $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ (Figure 2a) and continued: “I want an equation like this...We’re talking about positions, by saying operator $\hat{x}$...If I want to write an eigenvalue equation where that’s my operator, what is it going to tell me? What do these eigenvalue equations tell me, in general?” Dr. Holmes *identified and mapped (exported and imported) similar features of the concepts*, mapping “what these eigenvalue equations tell me” from the source to the target. He also *provided scaffolding for identifying and mapping objects and attributes* in the source for students to map to the target. They continued:

Dr. Holmes: If I operate on an eigenvector [points to $|E_n\rangle$ next to $\hat{H}$] with that operator, what is this [points to E_n in $\hat{H}|E_n\rangle = E_n|E_n\rangle$] then? [*Student:* A scalar]. What do I need here? [draws an arrow to the right of $\hat{x}$, writes “eigenvector”]. I need an eigenvector, and I need that same eigenvector here [writes “=”, leaves a space, draws arrow to a third space, writes “eigenvector”]. What do I need there? [points to space after “=”] (see Figure 2b).

Students: A scalar. The position.

Dr. Holmes: The position. But yeah, that’s the eigenvalue. [writes x_i after “=”] (Figure 2c).



(a) (b) (c) (d)

Figure 2. Dr. Holmes’s symbol forms written on the board during the Day 19 episode.

Dr. Holmes first *mapped the operator and eigenvector $\hat{H}|E_n\rangle$ objects* from the left side of the source eigenequation to the objects $\hat{x}$_ in the target, with the space labeled as “eigenvector.” In doing so, he *juxtaposed the operator and eigenvector symbols in the source and target*. This was supported by his *use of deictic gestures and inscriptions* (arrows) to highlight the correspondence between operator, eigenvector, and eigenvalue objects in the source and target eigenequations. He again *used deictic gestures and inscriptions,* saying “I need an eigenvector, and I need that same eigenvector here” and drawing arrows. This *mapped the relation* between the eigenvectors in the source eigenequations to the relation between those in the target eigenequation. By asking “what do I need to go there?” he directed the class’s attention to the eigenvalue in the source and to its position in the target. He affirmed the student responses. He *solicited student participation to map objects* and *juxtaposed eigenvalue symbols in the source and target* eigenequations.

Dr. Holmes continued, “Now we’re going to be in space space, as opposed to energy space or spin space, so you really have to keep these things straight in your head.” This *distinguished the attributes of objects in the source domain and target domain.* Dr. Holmes continued by next focusing on how to symbolize the eigenvectors in the position eigenequation they were creating:

Dr. Holmes: What should I use as how to represent an eigenvector of position? Like given some of the conventions we have for writing stuff [points to Figure 2a], what would be- [points to space for left eigenvector in target eigenequation]?

Students: x_i. x ket.

Dr. Holmes: Yeah, well first, it better look like that, right [writes kets $\hat{x}|\ \rangle = x_i|\ \rangle$, Figure 2c], and what do I want to put in here? [*Student:* x_i] x_i right? [writes $|x_i\rangle$, Figure 2d]. That’s fine. Because what the convention was, this is the eigenvalue for the eigenvector, right?

In asking students how to write a position eigenvector given conventions they had for writing spin and energy eigenequations, Dr. Holmes *used deictic gestures* to direct the class’s attention to the source eigenequations, *recalled the source attributes* of how eigenvectors are written, and *prepared students to identify similar attributes of eigenvectors in the source and target*. He also *solicited student participation to map the attributes of eigenvectors* from the source to the target. Considering the student responses, Dr. Holmes *exported source attributes* and *imported them to the target* that eigenvectors are symbolized as kets and labeled by corresponding eigenvalues. This *mapped the relation between eigenvectors and eigenvalues from the source to the target*. Dr. Holmes used a variety of pedagogical moves to engage the class in analogical reasoning as they created an eigenequation in their new position context, which we posit contributed to the class developing a unified understanding of eigenequations as they came to recognize them from three quantum mechanical systems as instantiations of the same overarching concept. Dr. Holmes used gestures and arrows to juxtapose $\hat{S}_z|\pm\rangle = \pm\,\hbar/2\,|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ from the source with $\hat{x}|x_i\rangle = x_i|x_i\rangle$ in the target. Together the class mapped objects (operator, eigenvectors, and eigenvalue) with their attributes and relations among the objects from the source contexts of spin and energy to the target context of position. His moves fostered the class's awareness of the shared properties among three distinct eigenequations, which expanded the common ground and contributed to the class's development of a unified understanding of eigenequations.

Conclusion

Engaging students in analogical reasoning can foster their unified understandings of a concept. Thus, we investigated the pedagogical moves a quantum mechanics instructor used to engage students in analogical reasoning. We identified eight pedagogical moves (Table 1) and described their potential to foster a class's development of unified understandings by expanding its common ground. Our characterization of pedagogical moves is the primary contribution of our study. The moves that stand out as most salient overall in our data set include juxtaposing the symbols that represent objects in the source and target domain, using deictic gestures, and using deictic inscriptions. These moves fostered the class's awareness of the shared structure among eigenequations in three different contexts and contributed to the class's development of a unified understanding of eigenequations. Given their productivity, we encourage instructors to use these pedagogical moves to analogize concepts and guide students to develop unified understandings.

Another contribution of this study is its connection of the theoretical constructs of analogical reasoning (Gentner, 1989), unified understandings (Lee & Heid, 2018; Zandieh et al., 2017), and common ground (Saxe et al., 2015). Studies have focused on individuals' unified understandings of mathematical concepts (e.g., Bagley et al., 2015; Cook et al., 2023; Melhuish et al., 2020; Serbin, 2023) but not yet on a class community's unified understanding. We posit that a class community can develop a unified understanding of a concept and thus expand its common ground by analogizing different instantiations of the concept. The class maps objects representing that concept, its attributes, and relations among that concept and others from the source to the target domain. Then, the target becomes part of the common ground, and the class comes to recognize those different examples as instantiations of the same overarching concept. When students engage in analogical reasoning, they can make connections between concepts, which is conducive to developing unified understandings of mathematical and physical concepts.

Lastly, we acknowledge some limitations to our methodology that could impact the interpretation of our findings. We cannot explicitly link the instructor's pedagogical moves with development of each student's unified understandings. Thus, we can only claim that his pedagogical moves related to analogizing were potentially conducive to guiding the class's shared unified understandings. It is outside the scope of this study to analyze individual students' unified understandings of eigenequations, so we suggest this as a direction for future research.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Number DUE-1452889. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

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