Norms and Values Upheld by Different Mathematical Activities: Proving, Defining, and Algorithmatizing

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Algorithms, proofs, and definitions all play important roles in mathematics as products of research and objects worthy of study in the classroom. Nevertheless, limited research has examined mathematicians' views of algorithms or compared their views across these objects and related activities. Based on interviews with nine discrete or computational mathematicians, we highlight similarities and differences between the values associated with these mathematical activities and relative importance placed on the varied activities.

Keywords: norms, values, proofs, definitions, algorithms

The mathematical community holds many views about what it means to do mathematics. While research has previously examined individual mathematicians' beliefs about what it means to do and teach mathematics (e.g., Rupnow, 2023; Woods & Weber, 2020), new attention has been focused on the values and norms of the mathematical community (Dawkins & Weber, 2017; Rupnow & Randazzo, 2023; Vroom et al., 2024). These works have focused on mathematicians' perceptions of how particular tools like proof (Dawkins & Weber, 2017) and definitions (Rupnow & Randazzo, 2023; Vroom et al., 2024) permit particular values to be upheld via norms of practice in the mathematical community. To build on this work, we examine a third tool, algorithms, and relate it to proof and definitions as we address the research question: How do norms and values around algorithms harmonize or suggest tension with norms and values around proving and defining?

Background Literature

Mathematicians engage in a variety of disciplinary practices that occur across subdisciplines of mathematics, including symbolizing, algorithmatizing, defining, justifying (Rasmussen et al., 2005), and computing (Lockwood et al., 2019). However, most examinations of these practices or of understandings of the corresponding objects (symbols, algorithms, definitions, proofs) have occurred in isolation. Algorithms and computing have received more attention recently because of their importance to computational thinking in mathematics and science (e.g., Lockwood et al, 2019; Weintrop et al., 2016), though algorithms have sometimes been viewed negatively as reinforcing procedural (as opposed to conceptual) understanding (Kamil & Dominick, 1997).

In contrast, extensive literature has examined the roles of definitions and proofs in mathematics education. Works focused on definitions have highlighted students' frequent use of extracted rather than stipulated definitions (e.g., Alcock & Simpson, 2011; Edwards & Ward, 2004; 2008), that is, self-created definitions extracted from known examples rather than stated definitions that pre-specify (stipulate) meaning. Research has also characterized the expectations of the criteria of a definition, such as existence and specification of necessary and sufficient conditions (Harel et al., 2006; van Dormolen & Zaslavsky, 2003) and pedagogical advantages of some definitions over others (Zazkis & Leikin, 2008). In addition to this extensive work on definitions, based on interviews with algebraists, Rupnow & Randazzo (2023) highlighted values and norms of the mathematical community related to definitions and the practice of defining.

Research attending to proof and proving is also myriad. Examples include examining different purposes that a proof can have (e.g., Weber, 2002), how different formats of instruction can align to different proof-focused pedagogical goals (e.g., Weber, 2004), different approaches to creating proofs in different mathematical subdisciplines (Dawkins & Karunakaran, 2016), how students understand proofs (e.g., Inglis et al., 2007; Stylianides & Stylianides, 2009), and differences in students' and mathematicians' approaches to creating proofs (e.g., Weber & Alcock, 2004). Additionally, Dawkins and Weber (2017) highlighted mathematicians' values and norms around proof production based on an examination of the published mathematics and mathematics education literature. Here we seek to compare and contrast mathematicians' views of algorithms, definitions, and proofs and the corresponding activity around them.

Theoretical Perspective

The Triadic Network of Justification (Laudan, 1984) is a theoretical framework that characterizes the conceptual connections among axiology, methodology, and factual claims in a community. Axiology relates to the ethics and values of a community, methodology relates to commonly accepted norms of practice in a community, and factual claims relate to what is accepted as settled fact in a community. While this framework was designed to conceptualize the nature of paradigmatic changes in the scientific community, this framework has been applied in mathematics education to characterize the values (axiology), norms (methodology), and theorems (factual claims) of the mathematical community with respect to proof (Dawkins & Weber, 2017) and definitions (Rupnow & Randazzo, 2023; Vroom et al., 2024).

Based on the existing literature (Dawkins & Weber, 2017) or interviews with mathematicians (Rupnow & Randazzo, 2023), these studies generated values upheld by specific norms related to proving or defining to characterize rationales behind practices of mathematicians. Vroom et al. (2024) used the defining norms and values to assist students in adhering to these expectations. Dawkins and Weber (2017) listed four proof-related values, each upheld by various norms: "mathematical knowledge is justified by a priori arguments", "mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author", "mathematicians desire to increase their understanding of mathematics", and "mathematicians desire a set of consistent proof standards" (p. 128). Rupnow and Randazzo (2023) highlighted two definition-related values, each upheld by various norms: "clarity in and for communication" and "freedom of choice for the use and creation of definitions" (p. 303). In this study, we build on the previously observed values and norms related to proofs and definitions while also using Laudan (1984) to conceptualize values and norms related to algorithms.

Methods

The first author emailed mathematicians at R1 institutions in the United States who were listed on their departmental website as researching applied/computational mathematics or combinatorics (or a similar description). That author then conducted 60-minute remote, screen-recorded interviews with nine such mathematicians. Five participants' research aligned more with applied/computational mathematics and four aligned more with combinatorics. All names are gender-neutral pseudonyms. Interview questions focused on participants' views of algorithms and their use, proofs and proving, definitions and defining, and how these areas interrelate.

Three researchers coded the interviews in MAXQDA 2020 (VERBI Software, 2019), with at least two researchers coding each interview. We entered coding with existing norms (codes) and values (categories) for proving (Dawkins & Weber, 2017) and defining (Rupnow & Randazzo, 2023) but were open to adding new codes and categories if needed. All three researchers initially open-coded two transcripts to create norm codes and value categories for algorithms in alignment with Laudan (1984). After discussing our open codes, we created a codebook for algorithm norms and added some new norms for proof. We then re-coded and came to consensus on the first two interviews. Finally, two researchers individually coded each of the remaining seven interviews and then discussed and came to consensus on the norms discussed. This process aligns with codebook thematic analysis (Braun et al., 2019) in that we entered analysis with some idea of the codes and categories (norms and values) we would use but were open to adding new codes as needed.

Results

We coalesced the themes (categories) we observed associated with algorithms in mathematics into three values:

1. Algorithms are useful for solving problems
2. Correctness of algorithms needs to be verified
3. Algorithms need to be clear to people and communicable to others.

Here we focus on the interplay between these values and the values for proving and defining in mathematics observed in the literature and our data. The first four of the following values for proving were introduced by Dawkins & Weber (2017), to which we added a fifth:

1. Mathematical truth is a priori
2. Mathematical knowledge and justifications should be a-contextual and independent of time and author
3. Mathematicians desire to increase their understanding
4. Mathematicians desire a consistent set of proof standards
5. Proving is socially mediated.

The following two values for definitions were introduced by Rupnow & Randazzo (2023):

1. Clarity in and for communication
2. Freedom of choice for the use and creation of definitions.

Below we highlight how these values arose in our data by discussing some of the norms (italicized) that upheld these values and then compare the values related to each mathematical practice to those in other practices.

Algorithm Values

Algorithms are useful for solving problems. The interviewees were unanimous, (n=9), in saying *algorithms are a tool*, although the interpretations of "tool" were myriad. For example, Becker took a high-level view on the definition of an algorithm and stated that "algorithms are a tool for us to systematically describe our way of thinking for a certain solution process." Similarly, Emery stated, "I'm thinking of an algorithm as meaning a procedure for accomplishing something," and explained that certain mathematical proofs can be viewed as algorithms by stating "there are constructive proofs which, in some sense, are inherently algorithms." On the other hand, Foster viewed algorithms more through the lens of a computational tool and stated, "Well, the purpose of an algorithm is always to compute something. Or to end up with something."

There was further consensus that *problems/contexts drive the use/creation of algorithms*. This norm conveys that there is a real-world problem that requires a solution, and this is the main motivating factor in algorithmic development. For example, as a researcher in graph theory and combinatorics, Emery stated that, "I mean, algorithmic research is devoted to finding procedures that will quickly answer certain questions." As a researcher in combinatorics, Harley stated that, "So I think that a problem is not really answered until you have some kind of algorithm that gives the answer. An object is not understood until you have an algorithm that constructs it." On the other hand, as a mathematical biologist, Ira stated that, "The goal is always to help me do a computation," with the implication that there is always some motivation from a biological problem for the algorithms required in Ira's work.

Correctness of algorithms needs to be verified. All interviewees observed that *deductive logic is important for algorithms*, as algorithms involve a sequence of steps and deductive logic is required for the algorithm to flow from one step to the next. For instance, Becker stated, "So algorithms are a way of describing patterns. Generally, what happens is you notice that certain things happen when you do things in a certain order", which highlights the importance of noticing and describing the steps within the algorithm. Devin made a similar comment but focused more on the big-picture sequence of subtasks within an algorithm: "What the algorithm does, the algorithm is going to allow you to compartmentalize what is happening, what is it going to, how is it getting there, and where are you going."

All but one of the interviewees suggested that *algorithms should be presented/tested with examples*. Emery viewed examples as a check on solutions: "Considering examples can be effective in understanding what an algorithm does." Ashton also highlighted the utility of examples for showing the purpose of steps in the algorithm: "You start with the simplest example you could possibly come up with and then just slowly ratchet up the level of complexity until you hit a point where—what should happen as you do that is that each step of the algorithm becomes clear why it's needed." As someone who has worked in a variety of contexts involving applied mathematics, Foster's experience indicated the importance of testing algorithms relative to examples with known solutions: "I never write an algorithm without testing it on things. Usually, you want to test it on things where you know the answer, and you want to see how well it's doing."

Three participants also highlighted that *using/creating algorithms drives research on convergence/stability properties*. Foster stated that:

> Sometimes we worry about [the fact that the algorithm does not satisfy all hypotheses of existing theorems for convergence] a little bit, and sometimes we just compute. And it depends on how confident we are that the algorithm is working. If we're pretty sure it works, we may not stop off and prove a theorem. If we have a little bit of a doubt, something is happening we're not expecting, then we might get into seeing what we could do about showing the algorithm really does converge to the answer that we're looking for.

Note that Foster sometimes will use an algorithm without proof of convergence, but this lack of rigor can drive future work for finding such a proof.

Algorithms need to be clear to people and communicable to others. All interviewees agreed that *algorithms need to be understood (enough, depending on context)*. Harley expected deep understanding, saying, "Everyone needs to have a conceptual understanding, a big picture, and a qualitative understanding." Others also conveyed expectations for understanding, such as Cooper, who highlighted the importance of using context:

> I've had these conversations also with students, and I ask them to show it in class where it's like, "Look, this fit is fantastic. If time was going the other way, no problem at all. But if you've got a fit and you want to predict the future and that fit is fantastic except for the last three weeks, then you look for something else, right?"

On the other hand, Ashton stated: "you can develop, use, and understand algorithms without even knowing that they actually truthfully do what you think they do… and just to be clear, sometimes it's separate because it's literally too hard" indicating that there can be situations, such as using algorithms in machine learning, where using algorithms as a black box is acceptable.

Algorithms and Proving

There are many similarities in the values common to proving and algorithms. Here we draw parallels between norms and values related to these practices. We first highlight the algorithm norm *deductive logic is important for algorithms* that upholds the value 'correctness of algorithms needs to be verified'. The argument can be made that every proof is an algorithm if one considers a sequence of logically justifiable steps starting from some assumptions and eventually reaching the desired outcome to describe both a mathematically rigorous proof and an algorithm. As noted above, all nine interviewees touched upon the importance of deductive logic for algorithms, and seven were coded for the proof norm *justification should be deductive and not admit rebuttals* that upholds the value 'mathematical truth is a priori'. Ashton stated, "a proof is an attempt to justify or argue that something is correct within a system," and Cooper stated that "If I've got a proof that is 99% correct in Mathematics, I've got nothing."

The development of both proofs and algorithms typically involves a period of inductive reasoning where various strategies are employed until either the proof is worked out in the former case, or the algorithm is constructed in the latter. The proof norm *proofs arise from inductive thinking but are written deductively* that upholds the value 'mathematical knowledge and justifications should be a-contextual and independent of time and author' was supported by seven interviewees. Foster said:

> I let them incubate for a while, and then I go to work on them more seriously and hope something drops. And I have techniques and machinery that I understand and can use, but you have to figure out which machine or which technique might be useful.

Harley stated that "…you try and see the pattern. And then you make a conjecture that this construction will produce the general object or will always produce whatever object you're looking for. And then you try to prove it. That's the hard part." Both quotes suggest the importance of looking for patterns (engaging in inductive thinking) as a precursor to writing a normative deductive proof.

While correct proofs and algorithms stand by themselves, a modification of the hypotheses of a theorem or a modification of the class of inputs to an algorithm can provide impetus for future work. The algorithm norm *problems/contexts drive the use/creation of algorithms*, upholding the value 'algorithms are useful for solving problems' is similar to proof norms of *proofs are tools for future problem solving* and *proofs lead to questions which lead to theorems (cyclic nature)* that uphold the proof value 'mathematicians desire to increase their understanding'. These proof norms were mentioned by seven and eight participants, respectively. For instance, Ashton stated, "Every proof is a possible technique you can apply in the future," Emery said, "You really try to get in and understand what the previous proof does in order to tailor it to new needs," and Harley remarked, "Generally, there will be lemmas and theorems that you can use, and you see how the tools have been applied and whether they can apply in your case." Notice these quotes all relate to the utility of knowing existing proofs for future proving.

Across the Three Practices

No matter the venue, anything mathematicians do needs to be clearly communicable to others, whether to other researchers or to students in the classroom. The algorithm value 'algorithms need to be clear to people and communicable to others', the proving values ‘mathematicians desire to increase their understanding’ and ‘proving is socially mediated’, and the defining value ‘clarity in and for communication’ all speak to the utmost importance that is placed upon the communication of mathematics to others. As Cooper states, “You got to know exactly what you’re talking about before you talk about it,” illustrating that there must be clear communication between mathematicians regarding agreed upon definitions. Harley stated, “there are proofs that are enlightening that really explain why something is true,” illustrating the point that mathematicians value proofs which clearly communicate the big ideas, over obfuscation of the important points.

Additionally, the utility of definitions, proofs, and algorithms are what imbues them with meaning. The algorithm value ‘algorithms are useful for solving problems’, the proving value ‘mathematicians desire to increase their understanding’, and the defining value ‘freedom of choice for the use and creation of definitions’ all support the idea that mathematics is more powerful when there is a larger aim in mind. Ashton illustrated the myriad utility of algorithms:

> Algorithms, to me, have-- okay, they have several purposes. One is to construct mathematical objects or examples. Another is as a proof technique. So often what you want to show is that a certain condition holds. And to do that, you construct an algorithm, prove that the algorithm terminates, and the existence of such an algorithm then demonstrates that the claimed result holds. And then a third would be just for computational purposes… I use them in all three of those ways.

On the topic of proofs, Harley discussed the utility of proofs by stating:

> So what are the purposes of proof? As I say, proofs are the binding cement or ligaments of mathematics. They’re what cement it into place, what make it solid or maybe the glue that holds it in place. Proofs are not in themselves the aim of mathematics, but they’re what hold it together. They’re what make it reliable.

Regarding defining, Ashton pointed out that definitions are in service of the mathematics, and the correct definition is important in order to be able to say something useful: “it depends on what definitions you need in order to be able to say something meaningful. So a lot of times, a theorem is only true if you restrict the definition in the correct ways.” Across these quotes, we see that algorithms, proofs, and definitions are not generally goals, of themselves, but rather tools to accomplish some broader purpose, such as solving problems, increasing understanding, or clearly specifying the limits of a claim so it can be proven.

Relative Importance of the Three Practices

Six of the participants intimated that algorithms, proofs and definitions are all equally important in mathematics. Ashton argued in this direction, stating:

> They’re all mutually supportive. So certainly, you can talk about each of those in isolation, and you should, and I do. But for me, I-- I really do want students to see that those are interrelated ideas. I mean, again, sometimes in order to define something, you have to define it by constructing it with an algorithm. And sometimes in order to prove something, you prove it with an algorithm. And sometimes what you’re proving is that an algorithm works. And sometimes in order to do that, you have to define a new thing. So these don’t have fixed roles in how they interact, right? They all interact in complicated ways and can show up in different spots.

Devin, an applied mathematician, similarly argued in this direction, stating:

> We have the clinical information, we have the model verification, and we have the theoretical proof. That’s the Holy Trinity, baby. That’s it. You are good to go. Man, you have it. And it works altogether. You need all of them working in tandem. Because a lot

> of people, they get the theoretical component, and they get the actual mathematical algorithms, and that's good. But what are these things helping you answer? They're helping you answer the real-world problems.

However, different participants expressed different views on the relative importance of the three practices in mathematics, with some interviewees pointing out that for many users of algorithms, the fact they are useful is the most important thing, not that they can be proved correct. Cooper drew on their own experience and stated, "To be a competently functioning professional, you don't need to be able to do a convergence proof like my numerical analysis colleagues. Because I've never done that for an algorithm, and yet my algorithms seem to be doing fine." Relatedly, some interviewees observed a disconnect between pure mathematicians, for whom proof is paramount, and their applied mathematics colleagues, for whom the utility of mathematical tools is more important. In response to the question of the purpose of proof in mathematics, Ira stated, "That's a loaded question. Proof is important in some branches of mathematics. It is unimportant in other branches of mathematics." In contrast, others viewed proof as more important than algorithms, such as Emery:

> I don't care so much about the algorithms except in the sense of finding such a proof, a constructive proof, which, again, is equivalent to an algorithm, but my focus is not usually on the algorithms, but rather on, again, on using them to prove something.

Still others highlighted the relative importance of definitions, such as Ira: "I tell students that clear, precise definitions are the glory of mathematics. They're the foundation. It cannot be overemphasized that a clear definition is extremely powerful."

Discussion

This study aimed to examine mathematicians' norms and values associated with the mathematical practices of algorithmatizing, proving, and defining. We observed three values related to algorithms, five related to proving, and two related to defining in our data and the literature. These included the new algorithm values of 'algorithms are useful for solving problems', 'correctness of algorithms needs to be verified', and 'algorithms need to be clear to people and communicable to others' as well as the new proof value of 'proving is socially mediated'.

Looking across the values related to the three practices, we observe similarities such as the importance of clear communication and the utility of the practice for consistency or useful applications. However, while many of our participants viewed all three practices as interrelated and mutually important, others strongly viewed some practice or practices as more important than others, potentially based on the subdiscipline of mathematics they were engaged in. Moreover, within each practice we also observe some tensions between different values, potentially related to whether the emphasis is on the process or resulting object. Proofs establish mathematical truth (given specified assumptions) beyond doubt, but proving is socially mediated and can change over time as conventions and expectations change. When defining a new concept, mathematicians have freedom to create a definition in the way that suits them, but definitions, once established, should remain consistent. Algorithms can be useful tools to solve problems, including problems that cannot reasonably be solved manually, but the process being followed in the algorithm needs to be understood, or at least verifiable to some degree, to be sure the tool is aiding in providing a true solution. Future research is needed to better understand which practices are more central to specific subdisciplines of mathematics and how to help students navigate these distinctions across different areas of mathematics and personal views of instructors.

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