Where is the Calculus in Calculus-Based Introductory Mechanics: A Textbook Analysis

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This paper describes a study of where and when calculus concepts and skills appear in the first semester of introductory physics. Following similar work in differential equations, engineering, and chemistry, we have adopted the Calculus Concept Framework and used the framework to identify which concepts and skills appear in each section of the first twelve chapters of a standard introductory physics textbook. Each section of each chapter was coded independently by three researchers and results were collated and discussed.

Keywords: physics, calculus in disciplines, textbooks

Calculus is a prerequisite for many courses in introductory physics. The sequence of courses taken by physical science and engineering students is frequently referred to as 'calculus-based' and has two or three semesters of calculus as pre- and co-requisites. (There is a parallel sequence of courses that do not require or use calculus, sometimes described as 'algebra-based' or 'trig-based.' These courses are generally taken by life science majors and pre-health professional students.) However, the nature of the relationship between calculus and introductory physics is undergoing renewed examination by scholars in both fields.

Introduction and Previous Research

There has been a significant body of work examining the use of mathematics in physics courses, for both introductory and more advanced levels. Redish and Kuo note that mathematicians and physicists use mathematics differently: while physicists load physical meaning onto symbols and equations, mathematicians tend not to (Redish and Kuo 2015). Thompson's quantitative reasoning framework describes the representation of measurable quantities with units (Thompson 2011). However, Biza and colleagues suggested that calculus in the disciplines could be a *filter* or *scaffold*: in the former case, calculus is used to determine who is qualified for later coursework; in the latter, it is useful as supporting knowledge to enhance success in later coursework (Biza et al., 2022).

Several prior studies have investigated student understanding of concepts and skills from calculus in introductory mechanics. (Here we focus on work in first-semester introductory mechanics, noting that there is a substantial body of relevant work in later courses.) For example, researchers have described a relationship between a limited understanding of definite integrals and difficulties in physics, in contexts including kinematics graphs (Beichner 1994, McDermott et al., 1987, Bajracharya et al., 2023). RUME scholars have reported that the conceptual understanding of integration that students need in physics is often not an outcome of standard calculus instruction (Orton 1983, Ferrini-Mundy & Graham 1994). Jones and colleagues have shown that most students leave calculus with a conceptualization of integrals as area under the curve (Jones 2015). However, more than half of the integrals in US physics textbooks used an adding-up-pieces conceptualization, and under 10% used area (Piña and Loverude 2019).

Several authors have reported that students perceive differences between practices in physics and calculus. Hitier and Gonzalez-Martin (2023) reported that "in mechanics, ready-to-use formulas are provided, so students can solve these problems without using knowledge from derivatives nor thinking in terms of covariation." Noah-Sella, et al. (2023) described a number of differences in the ways that students perceive related problems in physics and calculus, including the perception that physics problems require 'more thought' than calculus.

In the related field of engineering, faculty assessed the value of calculus "for mathematical maturity more than just the actual calculus" content, because "the way [the engineering course] is taught, you can do it without calculus." (Ferguson 2012, Faulkner, et al. 2019). Tague et al. (2013) collected data from engineering faculty on mathematical expectations and reported areas of priority and perceptions of students' competence. Areas of priority included evaluating solutions, using units and dimensions, knowing how to create and interpret graphs, performing algebraic manipulations, conveying engineering relationships mathematically, and using symbols instead of numbers. Faulkner et al. (2019) noted that faculty teaching engineering courses having mathematics courses as prerequisite or corequisite agree that their students have difficulties with: using and interpreting mathematical models; choosing and manipulating symbolic and graphical expressions; and, using computational tools. Participants identified students' struggles to "set up integrals and derivatives to describe physical situations" and described students' perception of calculus as just equations, preventing them from "capturing the underlying physical principles".

Scholars in several disciplines have posed the question of where exactly calculus is used and what calculus concepts and skills are used in courses for which calculus is pre-requisite. Many prior studies have adopted the Calculus Concept Framework [CCF] (Sofronas 2011), which was developed as the result of several interviews with experts. The CCF identifies a list of concepts and skills that are essential to calculus understanding and that are generally taught in a calculus course sequence (the concepts and skills are the first two sections shown in Figure 1, respectively). Czocher used the CCF to investigate what concepts and skills from calculus were used in a course in differential equations (Czocher et al., 2013). The work was replicated in additional contexts, including introductory engineering (Faulkner et al., 2020), and general chemistry (McAfee and Rodriguez, 2024). No similar analysis exists in introductory physics, which this study seeks to remedy.

Research question and methodology

For the sake of this study, we sought to investigate the research question: What concepts and skills from calculus are encountered by students in introductory calculus-based physics, and when are they encountered? For this initial effort, we made two choices to limit the scope of the project, allow for preliminary discussion of results, and establish a baseline for future research.

First, we chose to limit ourselves to the topics covered in the first semester of introductory physics, which is generally limited to classical mechanics. Most students take a sequence of introductory physics courses that are two- or three-semesters or a three- to five-quarters. Mechanics is generally the first course in the sequence, with electricity and magnetism, optics, and modern physics coming in subsequent courses. Some of these topics (notably electricity and magnetism) have different mathematical demands than the first-semester course and results for those will vary. In subsequent work we will extend this analysis to later topics.

Secondly, we chose to use the presence or absence of concepts and skills in the textbook assigned for the course as a measure. We recognize that focusing on the textbook is only a proxy for what students might realistically encounter in the course, as instructors might increase or decrease the emphasis on mathematical formalism in lecture or homework exercises. Future work will seek to characterize instructor practice as well and document what calculus concepts and skills students actually encounter in class and in homework.

We chose a standard physics textbook (Resnick et al., 2014). We chose to code chapters 2 through 12, which are the chapters generally covered in the first semester course. We omitted chapter 1, as it is purely a discussion of units and includes no calculus. We stopped after rotational motion and equilibrium. We excluded fluids or thermal physics, which are not covered in the first-semester course at our university but are sometimes included at others. The edition chosen was a matter of convenience; a later edition of the text is currently in use, but the older edition was more readily available. A quick comparison of selected chapters suggest that the newer edition would lead to nearly identical results.

We created a spreadsheet with the concepts and skills from the CCF and cells for each section of the chosen chapters for the first semester course. For early meetings, we chose a single chapter to code and the following week to compare results. When any of the three coders disagreed, we discussed the coding and how each coder was interpreting the prompt. While many codes were uncontroversial, some needed further discussion. Based on this discussion, we negotiated and clarified coding practices. For example, we had to decide whether the codes should reflect the use of concepts and skills in homework exercises at the end of a section, or just the narrative and sidebars. We chose to code based on both homework exercises and section text, noting that this might lead to over-estimating the experience of students. While a limitation, this also suggested avenues for further research that we are beginning to pursue.

We also observed that aspects of mathematical practice that were adjacent to calculus were present. As a result we added several extra categories that were not present in the CCF. While these are not calculus concepts and skills, they were clearly elements of the practice in using mathematics in the context of the physics textbook. These included the presence of graphs or figures, the use of function notation, the use of the vector concept and unit vector notation, and the explicit attention to units or dimensions. These should be considered supplemental to the main focus of calculus topics, but also fruitful topics of further study.

After a few weekly meetings focused on a single chapter, the researchers moved on to multiple chapters within a single week, and reconvening to assess how in agreement we were. As we came into alignment of definitions there was an increase in agreement with the topics covered in each chapter. By the end of the first-semester analysis, the agreement between coders very high. For example, of 243 cells in the Chapter 11 spreadsheet, all three coders independently coded the same result on 236 (97% agreement). Remaining disagreements generally included the presence of a concept or skill ‘hiding’ in a homework exercise.

Results

We describe the results in tabular form below. In the table, the numbers in each cell refer to the number of sections in which the concept or skill appears. Since chapters have different numbers of sections, the shading indicates frequency relative to the number of sections in a chapter, with solid green indicating a high fraction of incidence and solid red indicating none.

Below the table we make some observations and claims that summarize the results of our coding. Broadly speaking, in the calculus concepts and skills there are more red entries than green. The three most-frequently coded items from the CCF are arguably not even calculus: algebraic manipulations; facility with definitions and notation; and trigonometric manipulations.

Concepts	**Total Count (%)**	Ch2	Ch3	Ch4	Ch5	Ch6	Ch7	Ch8	Ch9	Ch10	Ch11	Ch12
Derivative	30.65	4	0	6	0	0	1	2	1	3	2	0
Integral	19.35	3	0	0	0	0	2	1	3	3	0	0
Limit	14.52	3	0	2	0	0	2	0	0	2	0	0
Approximation	8.06	1	0	0	0	0	0	1	2	0	0	1
Sequences/Series	3.23	0	0	0	0	0	2	0	0	0	0	0
Riemann Sums	6.45	0	0	0	0	0	0	0	1	2	0	1
Parametric and Polar Equations	0.00	0	0	0	0	0	0	0	0	0	0	0
Continuity	0.00	0	0	0	0	0	0	0	0	0	0	0
Optimization	0.00	0	0	0	0	0	0	0	0	0	0	0
Transformation	1.61	0	1	0	0	0	0	0	0	0	0	0
Skills												
Derivative Computations	6.45	4	0	0	0	0	0	0	0	0	0	0
Techniques of Integration	4.84	3	0	0	0	0	0	0	0	0	0	0
Limit Calculations	3.23	0	0	2	0	0	0	0	0	0	0	0
Sequences/Series	4.84	1	2	0	0	0	0	0	0	0	0	0
Area / Volume	14.52	1	3	0	0	1	1	0	2	0	0	1
Parametric Equations	0.00	0	0	0	0	0	0	0	0	0	0	0
Polar Coordinates	0.00	0	0	0	0	0	0	0	0	0	0	0
Trigonometric Manipulations	46.77	2	3	3	2	2	4	2	4	1	5	1
Epsilon Delta Proofs	0.00	0	0	0	0	0	0	0	0	0	0	0
Listening/Reading Comprehension	0.00	0	0	0	0	0	0	0	0	0	0	0
Manipulating Algebraic Expressions	98.39	5	3	7	3	3	6	5	9	8	9	3
Facility with Logarithms and Exponentials	0.00	0	0	0	0	0	0	0	0	0	0	0
Facility w/ Definitions & Notation	93.55	2	3	7	3	3	6	5	9	8	9	3
Additional Concepts & Skills												
Function notation	14.52	4	0	1	0	0	2	2	0	0	0	0
Something is a function of something	100.00	6	3	7	3	3	6	5	9	8	9	3
Units / dimensions	98.39	5	3	7	3	3	6	5	9	8	9	3
Vector concept	64.52	4	3	5	3	2	3	0	6	3	8	3
Unit vector notation	17.74	0	3	1	1	2	2	0	0	0	2	0
Graphs	33.87	5	3	7	2	0	1	1	1	1	0	0
Diagrams	82.26	3	3	5	3	3	4	4	8	6	9	3

Figure 1. The prevalence of calculus concepts and skills in the introductory physics textbook. The color indicates fractional prevalence, ranging from solid red (no occurrence) to solid green (present in all sections).

There is considerable variation in the frequency of calculus skills and concepts. The most prevalent calculus concept in our sample was the derivative. Derivatives appeared in 8 of the 11 chapters that we coded, and in 20 of the 62 sections (just over 30%). Integrals appeared in 5 of the 11 chapters, and 11 of 62 sections (just under 20%). In contrast, there was not a single epsilon-delta proof. Limits appeared in only four chapters (and 9 sections) and sequences and series in two sections of a single chapter. (The latter does become important in later courses, see Loverude 2024.) While the concepts of derivative and integral were somewhat prevalent, the skills of computing derivatives and techniques of integration were less frequently encountered. Even in calculus-based physics, derivatives and integrals are less-frequently encountered than algebra and trigonometry. Algebraic manipulations was coded for in every chapter, and 61 of 62 sections. Trigonometry was coded for in every chapter but one (chapter 2, one-dimensional kinematics) and appeared in 27 different sections.

The sequence of topics in introductory mechanics is mismatched to the sequence in calculus prerequisites, in terms of the use of calculus skills and concepts. Depending on prerequisite structures, students may be enrolled in first-semester calculus at the same time they are taking introductory mechanics. However, the most calculus-intensive sections of the text come very early. In the text, students encounter integrals in Chapter 2, which would come in the first or second week of the course. Integrals are covered later in the calculus sequence. Dot and cross products appear in first-semester physics not until Calculus III.

The calculus concepts and skills in use vary considerably across the semester. Not all chapters present the same level of calculus usage. Chapter two (one-dimensional kinematics) was one of the highest in terms of CCF concepts and skills coded, and chapter 4 had the highest prevalence of derivatives, though fewer limits and no integrals. The chapters on Newton's laws (5 and 6), in contrast, did not include a single limit, derivative, or integral. After chapter 2, students do not see another integral until chapter 7.

There are mathematical skills and concepts in physics that are not part of the CCF. While some of the skills and concepts identified in the CCF appeared rarely (or never appeared) in our analysis, we found that the mathematical practices were not fully captured by the CCF. As a result, we expanded our analysis to include additional categories that were relevant to the course but not included in the CCF: function notation, reasoning involving units, and unit vector notation, as well as the presence of graphs and diagrams. Notably, units and vectors were considerably more prevalent than core CCF concepts of limits, derivatives, or integrals. Units appear in nearly every section. Vectors are common as well, including dot and cross products.

Function notation is used early in the text, and then largely disappears. As the table shows, function notation is used in beginning sections, but the use of this notation diminishes, even though functions are present throughout the course. In later sections, function notation disappears entirely, even though expressions are complicated and identifying independent and dependent variables is not trivial. There is some reason to believe that different usage of function notation is a tension point for students in physics (Taylor and Loverude 2022).

Discussion

The results show that many sections of the introductory physics textbook do not use calculus skills or concepts, and the presence of calculus is not uniform. This raises some questions about the role of calculus prerequisites for this course, and whether those serve as scaffold or as filter. If calculus is to serve as scaffold, rather than simply a filter, instructors and textbook authors might wish to consider the mismatch between calculus and physics practices and the sequences of topics. We intend to compare our results with an analysis of texts that do not explicitly require calculus to see the extent to which these differ.

We note that the frequency of these concepts and skills in the textbooks might not reflect the experiences of students. Calculus concepts and skills appeared in derivations of equations or in certain examples and homework problems. Depending on instructor preferences, student experiences might vary greatly. Some students might see calculus in derivations but never need to evaluate a derivative or integral if relevant problems are not assigned. We have begun to probe the extent to which these data match the experiences of instructors and students at our university.

By focusing on the first semester, we have not included the significant presence of calculus (including multivariable and vector-valued functions) in the electricity and magnetism portion of the course. This will be another focus of future analysis.

Acknowledgments

The authors thank colleagues at CSUF and collaborating PIs Warren Christensen and John Thompson. This material is based upon work supported by the National Science Foundation under Grants PHY#1912660 and DUE#2336910; opinions, findings, and conclusions are those of the authors and do not necessarily reflect the views of the NSF.

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