Symbolic Forms for Inverse Relationships? Calling for Clarification on the Construct

Elizabeth Roan
Indiana University

Jennifer A. Czocher
Texas State University

Symbolic forms is a theoretical construct that has been used to study the meanings students have for symbols. One promising application of symbolic forms is to describe how students construct mathematical equations to represent real-world relationships, a skill needed across STEM disciplines. As more scholars use symbolic forms as an analytic tool across multiple contexts, they make adaptions to the theoretical construct by expanding existing lists of symbolic forms. In this paper, we continue this line of work by documenting undergraduate STEM majors' symbolic forms corresponding to inverse relationships that arose as they modeled dynamic scenarios. We end the discussion with theoretical considerations researchers must make to continue to use symbolic forms to describe students' equation construction.

Keywords: symbolic forms, inverse, differential equations, representations

Scholars across multiple disciplines are investigating how students understand and use symbols in a variety of contexts including chemistry (Becker & Towns, 2012; Rodriguez et al., 2018), physics (Dreyfus et al., 2017; Meredith & Marrongelle, 2008), and mathematics (Dorko & Speer, 2015; Jones, 2013). One thing these scholars have in common is their use of Sherin's (2001) theoretical construct of symbolic forms to operationalize how students understand patterns and meanings of symbols. One promising line of inquiry using symbolic forms is to describe students' construction of mathematical equations intended as representations of real-world scenarios (Schermerhorn & Thompson 2023; Roan & Czocher, 2024), a key skill needed across STEM disciplines. To work towards a fuller description of how students coordinate their knowledge of real-world relationships with symbolic forms to create mathematical equations, we used symbolic forms as a malleable tool of inquiry (see Moore et al., 2024). That is, we used symbolic forms to describe students' construction of equations to represent a predator-prey scenario and disease transmission scenario (Roan & Czocher, in press). That work allowed us to describe how students utilized symbol templates related to the mathematical concept of inverse, a type of mathematical concept that is important not only to modeling but to mathematics as a whole. We end the paper with theoretical considerations researchers must face to move forward in using symbolic forms as a theoretical construct to describe equation construction.

Theoretical Background and Literature Review

Symbolic forms (Sherin, 2001) are a theoretical construct used to describe how students think about equations. Symbolic forms are thought of as a blend (see Schermerhorn & Thompson, 2023) of two components: a symbol template and the associated conceptual schema. The symbol template is the pattern of the symbols. The conceptual schema associated with the symbol template is the idea being expressed by the symbols (Sherin, 2001). For example, a common symbol template used in mathematics is $\blacksquare + \blacksquare$. A conceptual schema associated with the symbol template $\blacksquare + \blacksquare$ is *a whole comprised of two parts* (Sherin, 2001). Conceptual schema are intended to be "relatively simple structures, involving only a few entities and a small number of simple relations among those entities" (Sherin, 2001, p. 491). The term conceptual schema is akin to diSessa's (1993) *phenomenological primitives* (p-prims), that is "elements of intuitive knowledge" like "increased effort begets greater results" (diSessa, 2018, p. 69). Further work has described the conceptual schema as the "underlying mathematical justification of the symbol template" (Schermerhorn & Thompson, 2023, p. 12). Conceptual schema correspond to the students understanding of relationships and structures (Sherin, 2001), thus conceptual schema may not align to normative characterizations of structures. At the heart of the theoretical construct is the hypothesis that "students learn to express a moderately large vocabulary of simple ideas in equations and to read these same ideas out of equations", where the "elements" of the vocabulary are symbolic forms (Sherin, 2001, p. 482). Researchers who use symbolic forms as a theoretical construct typically provide undergraduate students with pre-existing mathematical equations and ask them to interpret those mathematical equations. For example, Becker and Towns (2012) used symbolic forms to document how students understand mathematical equations in different physical chemistry contests like thermodynamics. Their findings indicate that students start to make sense of unfamiliar mathematical equations using heuristics that can be described by symbolic forms. In their study, the symbolic forms originally found in Sherin (2001) were sufficient to describe the students' reasoning. In other cases, researchers adapt the theory to include novel symbol template – conceptual schema pairs. For example, Dreyfus et al. (2017) investigated two students' sense-making while working on a quantum mechanics task. They used symbolic forms as an analytic tool to describe how the students made sense of the task. In doing so, they observed students using a symbol template – conceptual schema pair they called an eigenvector-eigenvalue symbolic form. Scholars have also used symbolic forms, commonly together with conceptual blending, to describe students' construction of mathematical equations (Ryan & Schermerhorn, 2020; Schermerhorn & Thompson, 2019, 2023). Schermerhorn and Thompson (2019) described how undergraduate physics students constructed a differential length vector for a spherical coordinate system. Part of this work was to add new symbolic forms to the list, and another part was to describe how students coordinated concepts from physics with symbolic forms to create their mathematical equations. They were able to identify what concept, like direction, motivated the student's use of a symbolic form, like parts-of-a-whole.

We situate our work within a similar line of inquiry as Schermerhorn and Thompson (2023); our work focuses on students' construction of differential equations that represent canonical scenarios students tend to encounter when in a differential equations class such as a predator prey scenario or a disease transmission scenario. A key mathematical relationship underlying those two scenarios is *inverse*. This is, because the predator – prey and disease transmission scenarios afford students the opportunity to think of flux to and from different populations (predator-prey) and/or sub-populations (disease transmission), students must coordinate their understanding of *additive inverse* with symbolic forms to create their models. To work towards a fuller description of how students coordinate their understandings of real-world relationships with symbolic forms to create mathematical models, we answer the research question: *what symbolic forms do STEM undergraduates use to represent inverse relationships between quantities that form their models for dynamic scenarios?* To answer this question, we share examples symbolic template conceptual schema pairs our participants used when representing scenarios that mathematically called for an additive inverse.

Methodology

The data for this paper draws from a larger dissertation study designed to characterize the cognitive obstacles students encounter while creating mathematical equations that represent dynamic scenarios. In this paper, we report on eleven participants' work on two modeling tasks: predator-prey and a disease transmission (Roan, 2024). The participants were high achieving undergraduate STEM majors who self-reported familiarity with mathematical concepts like instantaneous rates of change and who had taken or were enrolled in a differential equations course. The data were collected via task-based interviews conducted and recorded using Zoom. Images of the participants' written work were collected at the end of each session. In both tasks, participants were asked to create a (system of) differential equation(s) that would model the given dynamic scenario. In the predator-prey task, participants were asked to model the species dynamics between cats and birds in a back-yard habitat. In the disease transmission task, participants were asked to model how quickly a disease, like tuberculosis, spread through a community of infected (sick) or susceptible (healthy) individuals.

The goal of data analysis was to document distinct symbolic forms participants utilized while creating their models. We documented the symbolic forms by identifying the symbol template and identifying the conceptual schema associated with the symbol template. Identifying the use of a symbol template is straight forward, identifying the conceptual schema associated with the template was more inductive. This analysis was aided by the interview protocol, that guided the interviewer to ask the participant to explain why they chose a symbol template. For example, when a participant wrote down a mathematical equation like $C(t) \times B(t)$, the interviewer would ask "why did you decide to use $\times$ here?". We then used this information to infer what simple mathematical relationship (i.e., conceptual schema) the participant was trying to express using the symbol template. In instances where participants did not explicitly express their reason for using a symbol template, we inferred what simple mathematical relationship they were trying to represent using the participants' explanations of what the mathematical equation represented. We matched the symbolic forms our participants used when creating their models to clearly analogous symbolic forms documented in prior literature (see Brahmia, 2019; Dorko & Speer, 2015; Dreyfus et al., 2017; Jones, 2013; Pina et al., 2023; Riihiluoma et al., 2023; Schermerhorn & Thompson, 2023; Serbin & Uscanga, 2023). For example, our participants used the symbolic forms extensive – extensive, intensive – extensive, base $\pm$ change, and more, all of which are documented in Sherin's (2001) initial list of symbolic forms. We do not provide evidence of this use and instead focus on providing more details about the [potential] symbolic forms our participants utilized when creating models of dynamic scenarios that focus on representing inverse relationships. That is, we focus on the instances where our participants used symbol template – conceptual schema pairs that depict inverse relationships that did not have clear analogous symbolic forms from the literature. That is, the conceptual schema and symbolic template have not been reported together as a symbolic form previously. In this report, we show data excerpts from Ivory, Yixli, and Rillir, whose work was selected because they concisely explicated their reasoning with symbol templates. Ivory and Yixli were physics majors, and Rillir was an electrical engineering major.

Results

In the following section, we describe three symbol template – conceptual schema pairs our participants utilized when representing relationships that mathematically called for an additive inverse. We note here that, from the researcher's perspective, these relationships called for an additive inverse. We report the conceptual schema, the underlying mathematical justification, our participants associated with the symbol templates they chose to use, which might not align with the researcher's perspective. For each pair, we will provide the symbol template, our inferred conceptual schema, and evidence of that inferred conceptual schema. Additionally, we will state which task the symbol template – conceptual schema pair was utilized for, as not all pairs were utilized across all tasks. An overview of our findings can be seen in Table 1.

Table 1 An overview of symbolic forms

Symbol Template	Conceptual Schema	Task Scenario
$x = \frac{1}{y}$	Quantity y is inversely related to quantity x.	Disease Transmission
$x = -y$	Flux in equals the flux out	Disease Transmission
$x = 1 - y$	Quantity x is the percent complement of quantity y.	Predator - Prey Disease Transmission

Inverse

Ivory (physics) was working on the disease transmission task. She had established that the instantaneous rate of change of the sick population would be represented by the symbol $\frac{dS}{dt}$ and the instantaneous rate of change of the healthy population would be represented by the symbol $\frac{dH}{dt}$. Ivory sought to create a mathematical equation that would represent the relationship between the instantaneous rate of change of the sick and healthy populations. To meet her goal, Ivory wrote down $\frac{dS}{dt} = 1 \big/ \frac{dh}{dt}$. When explaining her model, Ivory stated

> Ivory: You could also do one over $\frac{dh}{dt}$, I think. [LAUGHS] Because you're either sick or not sick. So the inverse of sick people would be the healthy people. And the-- the inverse of healthy people would be the sick people.

A few moments later, when asked about her model she explained again that she had written $\frac{dS}{dt} = 1 \big/ \frac{dh}{dt}$ because "it's just, you're either healthy or you're sick." We infer from Ivory's explanation that she envisioned that the entire population could be split into two subpopulations: the sick and the healthy. Further, if a person is sick, then they cannot be healthy; thus her logic goes, the sick population and the healthy population are inversely related.

The conceptual schema, the intuitive mathematical relationship being represented, is: one quantity is inversely related to another quantity. The symbol she associated with this symbol template was $x = \frac{1}{y}$. Mathematically, we would anticipate that the conceptual schema associated with such a symbol template would be something like one quantity is inversely proportional to another quantity. However, it is not the case that Ivory attended to proportionality when writing down her mathematical equation, leading us to believe that Ivory's conceptual schema paired with the symbol template $x = \frac{1}{y}$ is novel, though nonnormative. In fact, Ivory edited her mathematical representation and changed the symbol template multiple times before settling on a normatively correct mathematical representation for an SI model, $\frac{dS}{dt} = -\frac{dH}{dt}$. We will discuss the ramification of her edits in the discussion.

Additive Inverse

Like Ivory, Yixli (physics) was also working on the disease transmission task. He had previously created a mathematical equation that represented the change in the infected population, which he labeled as ΔI. A full breakdown of Yixli's model is given below in Table 2.

Table 2 A breakdown of Yixli's mathematical equation for the change in the infected population

Quantity	Symbol / Equation
Infected people	$I(t)$
Susceptible people	$S(t)$
Possible Encounters	$I(t) \times S(t)$
Exposures per Possible Encounter	α
New infection per exposure	τ
Change in the infected population	$I(t) \times S(t) \times \alpha \times \tau$

Yixli sought to create a mathematical equation that would represent the instantaneous rate of change of the susceptible population. To meet his goal, Yixli wrote $\Delta S = -I(t) \times S(t) \times \alpha \times \tau$. When explaining his model, Yixli explained that

> Yixli: OK, so I think this will just be inverted? [writes $-\tau \times \alpha \times S(t) \times I(t) = \Delta S$]. And that's it. You know, because if we're- if we gain two on the-the infection, then we lose two on the susceptible. Um, so it's just going to be the negative reciprocal. Not negative reciprocal, but, you know, um, this is going to be the negative of- of- of the rate of the infection.

We infer from Yixli's explanation that he envisioned that flux out of the susceptible population is equal in magnitude to the flux into the infected. That is, the conceptual schema being represented here is: flux in equals the flux out. The associated symbol template to this conceptual schema is $x = -y$. We used Yixli as an example because of his well-worded explanation, but many of our participants used this symbolic form during the disease transmission task.

Percent Complement

Rillir was working on the predator-prey task. He had previously created a mathematical representation for the number of interactions between a cat and a bird that do not result in death. The quantities and their corresponding symbols within his mathematical equation are shown in Table 3.

Table 3 A breakdown of Rillir's mathematical representation for the number of interactions between a cat and a bird that do not result in death.

Quantity	Symbol/Equation
Bird population	$B(t)$
Cat population	$C(t)$
Total possible interactions between species	$B(t) \times C(t)$
Percent of possible interactions that happen	α
Percent of actual interactions that do not result in death	β
Number of interactions between a cat and a bird that do not result in death	$B(t) \times C(t) \times \alpha \times \beta$

Rillir sought to create a mathematical equation representing the number of interactions between a cat and a bird that result in death. To meet his goal, Rillir wrote down $100 - \beta = G$, explaining that "since it's a β. Since it's a percent. You can do $100 - \beta$". He then replaced β with G in his mathematical representation so that $B(t) \times C(t) \times \alpha \times G$ represented the number of interactions between a cat and a bird that do result in a death. We infer from Rillir's explanation that he thought that since there are two outcomes of an interaction, death and not death, the percentage of interactions that do result in death must be the percent complement of the percentage of interactions that do not result in death. That is, the conceptual schema being presented here is: one quantity is the percent complement of the other. The associated symbol template to this conceptual schema is $1 - x = y \; or \; 100 - x = y$, depending on whether the percentage is in decimal form or not. One other participant used this conceptual schema-symbolic template pair, and they also explained their mathematical representation in a similar way to Rillir.

Discussion

In this paper, we have presented three conceptual schema–symbol template pairs our participants utilized while modeling dynamic scenarios that have some connection to the idea of *inverse*. The pairs were observable because the dynamic scenarios the participants modeled lent themselves to consider flux within a population and between populations. Our work expands what is known about how students use of symbolic forms to represent inverses and also has implications for studying students' mathematical equation construction using symbolic forms.

Serbin and Uscanga (2023) used the symbolic form construct to discuss how pre-service mathematics teachers reasoned about additive, multiplicative, and compositional cancellation. The symbolic form construct focused their analysis on what kinds of symbolic templates (i.e., templates with or without explicit depictions of inverse or identity) their participants used when thinking of cancellation operations. We infer that Serbin and Uscanga's descriptions align with two of the three ways students' reason about inverses across contexts described by Cook et al. (2023): inverse as an undoing and inverse as a manipulated element. By studying students' construction of mathematical equations that represent dynamic scenarios, we have described three conceptual schema–symbol template pairs that align with the third way of reasoning Cook et al. (2023) described: inverse as a coordination of the binary operation, identity, and set. Our efforts add to the growing list of symbolic forms and further test the symbolic form construct in a new area and adding to its applicability in more contexts.

However, we have rhetorically referred to these as conceptual schema – symbol template pairs because there are alternative interpretations of each of these pairs that brings to light considerations for using symbolic forms to describe equation construction. First, we proffer alternative interpretations of each example and second, raise the difficulties of making inferences based on student reasoning.

First, Ivory used the conceptual schema "quantity y is inversely related to quantity x." paired with the symbol template $x = \frac{1}{y}$. She, in that moment, justified using symbol template $x = \frac{1}{y}$ using the mathematical idea of inverse. When students create mathematical equations to represent real-world relationships, the "underlying mathematical justification" (Schermerhorn & Thompson) for the symbol template may not be mathematically accurate. In prior literature, researchers have discussed how the conceptual schema might not accurately depict the phenomena of interest (see Schermerhorn & Thompson, 2023) but the schema and template still aligned mathematically. This raises the question, do cases where the conceptual schema and the symbol template do not mathematically align belong on the growing list of symbolic forms seen across contexts? We assert that they should belong on the list, because these cases are informative about what underlying justifications students associate with the symbol templates. Further, these indicate places where students might need support to disentangle nonnormative conceptual schema with symbol templates. Ivory later edited her model because she realized that her mathematical equation was incorrect and that the conceptual schema she had associated with the symbol template $x = \frac{1}{y}$ did not match the relationship between her quantities as well as she originally thought. When students make edits to their mathematical equations the conceptual schema may change. When that change occurs, researchers must consider if theoretically that is that another instance of a symbolic form or a different manifestation of the same symbolic form.

Second, we consider Yixli's use of the conceptual schema *flux in equals the flux out* along with the symbol template $x = -y$. An alternative interpretation of Yixli's example is that he used the symbolic form Sherin (2001) called competing terms, where the conceptual schema was *influences in competition*, and the symbol template was $\blacksquare \pm \ldots \pm \blacksquare$. In Yixli's case, he simply had one term that was doing the influencing. From our perspective, this interpretation overlooks that Yixli created a mathematical equation that simultaneously described a positive influence on one quantity and a negative influence on another, a key aspect of the justification Yixli gave for using that symbol template. Thus, his conceptual schema was distinct from *influences in competition* since, from his perspective, no influences were competing.

Lastly, we consider Rillir's use of the conceptual schema *quantity x is the percent complement of quantity y*, along with the symbol template $x = 1 - y$. In our interpretation, the $1 - y$ or $100 - y$ is like an operator, a thing done to a percentage to get its complement. This aligns with other explanations of symbolic forms such as in Jones (2013), who described the parts of an integral as operators that acted on an original function. An alternative interpretation of Rillir's example is that he instead activated the whole - part symbolic form (Sherin, 2001) where the conceptual schema is taking away a part of an original whole and the symbol template is $[\blacksquare - \blacksquare]$. Sherin (2001) described this symbolic form as "a new net amount is produced by taking away a piece of an original whole" (p. 534). It is unclear the extent to which Rillir and other participants saw the 1 or 100 in the symbol template as a whole that the percentage is being taken from rather than an operator that does something to the percentage. When cases like this occur, researchers must decide if theoretically this is an instance of a singular symbolic form or an instance where the student is using a few symbolic forms waring a trench coat. Researchers need to take into consideration the loss or gain in explanatory power when making that decision. We infer the theoretical consideration is directly linked to the grainsize of students' justifications. Thus, researchers must consider the theoretical construct's ability explicate differing grain sizes of the mathematical justifications underlaying students' choice in symbol template. We acknowledge some of our difficulty classifying our participants symbol template – conceptual schema pairs as symbolic forms may lie with the interview protocol. However, our limitations to making more definitive claims have brought to the forefront important theoretical considerations. That is, the three examples reported in our paper bring to light considerations research must address if work is to move forward in using symbolic forms to describe students' construction of mathematical equations.

Acknowledgments

The research reported in this paper is supported by National Science Foundation Grant No. 1750813 with Dr. Jennifer A. Czocher as principal investigator and by Doctoral Research Support Fellowship given by Texas State University.

References

Becker, N., & Towns, M. G. (2012). Students' understanding of mathematical equations in physical chemistry contexts: An analysis using Sherin's symbolic forms. *Chemistry Education Research and Practice*, *13*, 209-229.

Brahmia, S. (2019). Quantification and its importance to modeling in introductory physics. *European Journal of Physics*, *40*.

Cook, J. P., Richardson, A., Strand, S., Reed, Z., & Melhuish, K. (2023). Examining the concept of inverse: Theory-building via a standalone literature review. *The Journal of Mathematical Behavior*, *72*, 101100.

diSessa, A. (2018). A Friendly Introduction to "Knowledge in Pieces": Modeling Types of Knowledge and Their Roles in Learning. In (pp. 65-84). https://doi.org/10.1007/978-3-319-72170-5_5

diSessa, A. A. (1993). Toward an Epistemology of Physics. *Cognition and Instruction*, *10*(2/3), 105-225. http://www.jstor.org/stable/3233725

Dorko, A., & Speer, N. (2015). Calculus Students' Understanding of Area and Volume Units. *Investigations in Mathematics Learning*, *8*, 23-46.

Dreyfus, B. W., Elby, A., Gupta, A., & Sohr, E. R. (2017). Mathematical sense-making in quantum mechanics: An initial peek. *Physical Review Physics Education Research*, *13*.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior*, *32*, 122-141.

Meredith, D. C., & Marrongelle, K. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics*, *76*, 570-578.

Moore, K. C., Stevens, I. E., Tasova, H. I., & Liang, B. (2024). Operationalizing Figurative and Operative Framings of Thought. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's Genetic Epistemology for Mathematics Education Research* (pp. 89-128). Springer International Publishing. https://doi.org/10.1007/978-3-031-47386-9_4

Pina, A., Topdemir, Z., & Thompson, J. R. (2023). Student Understanding of Eigenvalue Equations in Quantum Mechanics: Symbolic Forms. 25th Annual Conference on Research in Undergraduate Mathematics Education, Omaha, NE.

Riihiluoma, W., Topdemir, Z., & Thompson, J. R. (2023). Modifying symbolic forms to study probability equations in quantum mechanics. 25th Annual Conference on Research in Undergraduate Mathematics Education, Omaha, NE.

Rodriguez, J.-M. G., Santos-Diaz, S., Bain, K., & Towns, M. H. (2018). Using Symbolic and Graphical Forms To Analyze Students' Mathematical Reasoning in Chemical Kinetics. Journal of Chemical Education, 95(12), 2114-2125.

Roan, E. (2024). Explaining the Mathematical Expressions Undergraduate STEM Majors Construct to Model Dynamic Scenarios. [Doctoral dissertation, Texas State University]

Roan, E., & Czocher, J. A. (2024). The Cognitive Obstacles Associated with Structuring for Mathematization Undergraduates Encountered During Dynamic Modeling Tasks. 26th Annual Conference on Research in Undergraduate Mathematics Education, Omaha, NE.

Ryan, Q. X., & Schermerhorn, B. P. (2020). Students' use of symbolic forms when constructing equations of boundary conditions. *Physical Review Physics Education Research*, *16*.

Schermerhorn, B. P., & Thompson, J. R. (2019). Physics students' construction of differential length vectors in an unconventional spherical coordinate system. *Physical Review Physics Education Research*, *15*.

Schermerhorn, B. P., & Thompson, J. R. (2023). Making context explicit in equation construction and interpretation: Symbolic blending. *Physical Review Physics Education Research*, *19*(2), 020149.
Serbin, K. S., & Uscanga, R. (2023). "Just Get Rid of It:" Students' Symbolic Forms for Cancellation. 25th Annual Conference on Research in Undergraduate Mathematics Education, Omaha, NE.
Sherin, B. L. (2001). How Students Understand Physics Education. *Cognition and Instruction*, *19*(4), 479-541.