Students' Connections About Equivalence Across Contexts: Substitution and Known vs. Own Equivalences

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The concept of equivalence is fundamental to problem-solving in mathematics. Particularly, an oft-utilized feature of equivalence is substitution, in which the mathematics doer replaces one object with an equivalent object. Equivalence in general, and substitution in particular, have been identified as common threads that can tie together different areas of mathematics, but do students notice these connections in their own mathematics? In this paper, I report on task-based clinical interviews with students exploring the commonalities they identify among tasks implicitly involving equivalence relations. In particular, I discuss themes that students explicitly identified as relevant when answering the interview tasks: (1) substitution equivalence and (2) a distinction between students using a known equivalence for substitution or introducing their own equivalence.

Keywords: equivalence, substitution, actor-oriented, student-generated connections

Equivalence has two fundamental uses throughout mathematics: it is used to express the *sameness* of mathematical objects as well as their *interchangeability*. In fact, Liang (2021) proposed that equivalence "can be used as a thread to connect mathematical concepts and topics," but that much "remains to be done to analyze the nature of equivalence and substitution in different contexts, and in developing suitable tasks to provide students […] with opportunities to enrich their understandings of these ideas" (Liang, 2021, p. 43). Studies have extensively examined the substitutive aspect of equivalence with respect to equality, but it remains to be seen how the substitutive aspect of equivalence is conceptualized by students when working with non-equality equivalence relations, and moreover if students see these ideas as related across contexts. Thus, for this study I sought to answer the following research question: *What connections do undergraduate students identify among tasks involving equivalence relations?*

Background & Theoretical Framework

Actor-oriented Perspective

To examine whether students identified any commonalities across tasks involving equivalence in various forms, I adopted the actor-oriented perspective. This perspective was appropriate because my purpose was to "[inquire] into what types of common features students might perceive across cases" (Ellis et al., 2007, p. 227). Under this perspective, Lockwood (2011) defines a *researcher-generated connection* as a connection inferred by the researcher, i.e., "a connection that was not verbally articulated or borne out in some external activity" (p. 311) by the student, and a *student-generated connection* is one in which a student explicitly articulates or demonstrates that they view a particular connection. Student-generated connections are important to study because they are the connections that students are consciously aware off, thus students may more intentionally attend to or rely on them in their problem solving. In short, this framework provides researchers with an opportunity to examine what *students* see as relevant when identifying similarities between their current and past mathematical activity. Lockwood (2011) characterizes types of student-generated connections as:

- *elaborated* or *unelaborated* connections, where the former refers to connections in which the student explains what prompted them to draw a connection, and the latter refers to when students mention a connection but do not expound upon what led them to make this connection;
- *conventional* or *unconventional* connections, where the former "occur when students make connections to problems that may conventionally be considered isomorphic to the problem at hand" (p. 311), and the latter "occur when students relate situations that an expert or observer might not find mathematically isomorphic" (p. 312); and
- *referent type*, meaning the types of objects students draw connections to, including particular problems, problem types, and strategies or techniques for solving problems.

My work builds upon the existing work of Cook et al. (2022) and Richardson et al. (2024)—who presented *researcher-generated* views of the key commonalities in reasoning about equivalence across contexts—by examining students' views on the commonalities they attend to and identify when reasoning about equivalence across contexts.

Substitution Equivalence

Researchers have posited that a desirable conception of equality is a relational one, i.e., a recognition that equal mathematical objects have the same value (e.g., Baroody & Ginsburg, 1983; Jones & Pratt, 2012; Kieran & Martínez-Hernández, 2022). An elaborated, *full relational* conception of equality has been proposed as more completely capturing the nature of equivalence, which involves recognizing the sameness of mathematical objects and that one object can be substituted for an equivalent object (Carpenter et al., 2003; Jones et al., 2012; Jones & Pratt, 2012; Molina et al., 2009; Tirpáková et al., 2023). In particular, for the symbolic representation $x = y$, the sameness component would manifest as x "is the same as" y and the substitutive component would manifest as x "can be exchanged for" y (Jones, 2008, p. 154). I use the *sameness* and *substitutive* aspects as similarly applying to equivalence, whereby objects related by some equivalence relation have the properties that they are the same with respect to some property of interest and they can be freely substituted for one another.

Tirpáková et al. (2023) elaborated on the idea of substitution equivalence by introducing the idea of students' *own equivalences* to describe when students define an arbitrary equivalence for the purpose of using substitution in mathematics. This is contrasted with what I call *known equivalences*, which are equivalences that students do not define themselves and are instead recalled, proven to be true, or created via a sequence of appropriate transformations. For example, substitution of 0.5 with $\frac{1}{2}$ in simplification of the expression $\frac{2}{3} + 0.5 + \frac{3}{4} - \frac{4}{9}$ would be the use of a *known equivalence* because it is based on the numeric equality of the objects in question. Contrarily, using substitution of variables $u = 2x$ to rewrite the integral $\int_0^{\sqrt{\pi}} 2x \sin(x^2)\, dx$ as $\int_0^{\pi} \sin(u)\, du$ would be considered an *own equivalence* because it is an arbitrary, albeit convenient, replacement introduced by the mathematics doer and is not based on some known, given or discovered property of the given mathematical objects.

Introducing an *own equivalence*, such as in the integration example above, is important because it involves attending to the structure of the given problem without changing its content mathematically. Tirpáková et al. (2023) emphasize the importance of students' own equivalences as they can be introduced strategically to make the intended calculations more efficient or easier to perform. Moreover, Tirpáková et al. (2023) assert that "perceiving structure of the given equation allows the students to effectively connect their various knowledge of mathematics by using appropriate substitution," and that by defining their *own equivalences*, students "can transform an unknown task into a task that they are already able to solve" (p. 486).

While the idea of substitution has been studied extensively as it relates to equality of expressions, this work has been done primarily at the level of researcher-generated connections, and it remains to be seen what findings are also applicable in more general instances of non-equality equivalence. Thus, my work addresses a gap in the literature by investigating the commonalities students identify in how they view equivalence (in particular, substitution equivalence) across contexts.

Methods

Data Collection and Task Design

I conducted task-based clinical interviews (Clement, 2000) with 16 students who were either currently enrolled in or had recently completed the last course in a three-course calculus sequence. Each student participated in an individual interview lasting between 90 and 120 minutes. Students' written work was recorded on a digital whiteboard iPad application and synced with an audio recording of the interview.

I selected this population of students because they would have had prior exposure to the mathematics required to complete each interview task, but they would not yet have been exposed to formal notions of equivalence relations. I conducted task-based clinical interviews because the purpose of a such an interview is to capture students' genuine interactions with the task setting in order to understand how they reason about a concept in the moment. My goal was to learn about students' current understandings of mathematical ideas—with the hope that the given tasks would elicit students mentioning ideas of equivalence or substitution—without deliberately influencing changes in students' thinking.

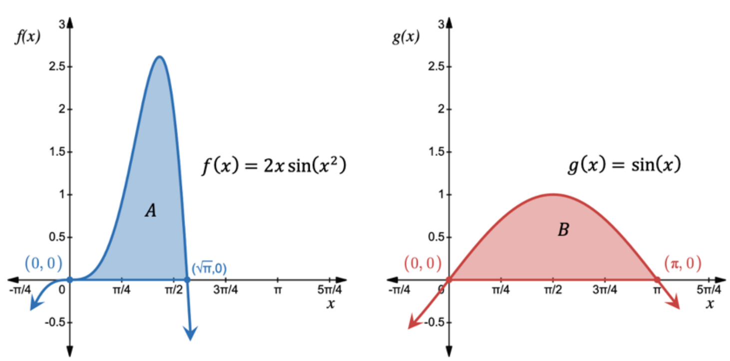
Task 1:	Simplify the following expression: $\frac{2}{3} + 0.5 + \frac{3}{4} - \frac{4}{9}$
Task 2:	Simplify the following expression: $\sin(2400°) + \cos(1110°) + \tan(585°)$
Task 3:	Show that the two shaded areas in the graphs below are equal: 
Task 4:	Solve the following equation on the set of real numbers: $\frac{x\sqrt[3]{x}-16}{\sqrt[3]{x^2}-4} - \frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}+1} = 17$

Figure 1. Tasks administered during the interviews.

The interview tasks (Figure 1) were selected because they could each be completed by using some form of substitution to make the given task simpler to execute, but the tasks involved different contexts and slightly different notions of equivalence. For example, in Task 1, each term can be substituted with an equivalent fraction so that every term has a common denominator, and each original term is numerically equal to its substituted term (e.g., $\frac{2}{3}=\frac{24}{36}$). In Task 2, each angle can be substituted with its coterminal angle in the interval $[\mathbf{0}, \mathbf{2\pi})$, and while these angles are not numerically equal (e.g., $\mathbf{585}° \neq \mathbf{225}°$), they are equivalent in the sense that they share the same position on the unit circle and thus would produce the same value when used as the argument for given trigonometric functions. The interview tasks also involved different notions of substitution. For example, the substitution discussed above for Task 1 would be considered using a *known equivalence* because it involves substituting one number for another which is known to be equal. Contrarily, the example of introducing $\boldsymbol{u}$-substitution to rewrite an integral (discussed in the "Substitution Equivalence" Section) for Task 3 would be considered using an *own equivalence*.

Data Analysis

The audio for each interview was transcribed verbatim and then coded using Clement's (2000) cycles of interpretive analysis. I focused on ideas that students mentioned repetitively throughout the interview and across tasks because I wanted to investigate what features of the tasks and their solutions that students deemed as relevant across a variety of contexts. In particular, I was interested in whether the students identified anything pertaining to equivalence.

I distinguished between what I deemed as common ideas that students appeared to recognize across tasks (*researcher-generated connections*) and what students *explicitly* mentioned as being common or similar across tasks (*student-generated connections*). That is, instances labeled as researcher-generated connections could have been connections that the students *implicitly* recognized but did not *explicitly* identify as being connections across tasks in their utterances or writing, i.e., there was not sufficient evidence to show that students did in fact see these as connections across tasks. Thus, connections classified as researcher-generated have the potential to be commonalities that students identify across tasks but are simply not made explicit by the students. In the results, I provide examples of explicit, student-generated connections; I used Lockwood's (2011) framework for categorizing the types of student-generated connections. While I was interested in all types of connections that appeared in the data, in the results I provide two examples of elaborated, conventional student-generated connections.

Results

I will now discuss two particular connections that students identified during their interviews. For brevity, I discuss selected excerpts from two participants: Jackson and Alex (pseudonyms). I consider these to be prototypical examples that are illustrative of the commonalities identified across contexts by multiple interview participants: *substitution equivalence* as a problem-solving strategy and the difference between *known* and *own equivalences*.

Student-generated Connection: Substitution Equivalence

Jackson identified his use of *substitution*—i.e., replacing one mathematical object with an equivalent one—as part of his strategy for answering several of the interview tasks. As evidence, Jackson continually referenced how he would "exchange" or "interchange" equivalent elements for the purposes of changing the given task into a "more convenient" format throughout multiple interview tasks. In particular, he expressed that he viewed equivalence as a necessary condition for being able to freely substitute mathematical objects.

In his work on the first task, Jackson simplified the expression by first writing each term as a fraction, then finding common denominators pairwise and simplifying left-to-right. He wrote 4/6 and 3/6 in place of 2/3 and 0.5, respectively. He then justified this step of his work by saying:

Jackson: That ratio would be the same if you had 6 things and you picked 4 out of that. It's like the same ratio, but 4/6 was just more convenient to work with than 2/3 is. **And since they're the same thing, you can just interchange them.**

That is, Jackson used the idea of substitution as a solution strategy because he "interchange[d]" the fractions 2/3 and 4/6 and expressed that he knew this was allowed mathematically because "they're the same thing."

On the second task, Jackson subtracted the highest multiple of 360° that he could from each angle to convert into a coterminal angle between 0° and 360°. He then used the unit circle to read off or calculate the desired trigonometric values. When asked to justify his work, Jackson explained:

Jackson: So if it's anything more than 360 you can just do, um, your starting number minus 360 because that'll take out the unnecessary rotation and then just give you the exact, like angle that you're looking for because **if it was 30 or if it was 390 those are the same thing because they both lead to the same spot.** […] And then whatever I'm finally left with, that's what gives me my angle that I can use.

He then clarified that he viewed *sameness* of angles in terms of their position on the unit circle, and expressed that this sameness allowed him to substitute one angle for another for the purposes of this task:

Jackson: **For two angles to be the same on the unit circle they have to have the same end coordinates.**

Interviewer: Okay. […] So as long as two angles […] have the same end coordinates, what happens?

Jackson: **Then they can be exchanged** […] within trigonometric functions.

Similar to the first task, on the second task Jackson explained his process of "exchang[ing]" one object with an equivalent one that would yield the same end result as the original presentation of the task. In the first task this equivalence condition was numeric equality, whereas in the second task this equivalence condition was position on the unit circle. However, in both cases, Jackson expressed that this equivalence condition allowed him to substitute one element for an equivalent one and still preserve the solution to the original task. Moreover, Jackson was careful to clarify that his equivalence condition for Task 2 allowed him to exchange angles specifically "within trigonometric functions."

Until this point, I had viewed Jackson's use of substitution as a *researcher-generated connection*, because although Jackson was continually mentioning the same ideas of using substitution across tasks, he did not explicitly state that he did in fact view these uses of substitution as being instances of one unified idea. However, after completing all the interview tasks, Jackson summarized his work by saying, "**If two things are equal, then you can substitute them**, or like interchange them with whatever's more convenient." He expressed that in hindsight he recognized that he used substitution on every interview task:

Jackson: But like, if it's the same thing, you can exchange it with something else and that makes it a lot easier. And that's kind of like one of the core [ideas] of math I guess because you use it like a lot more than you would think. Like **I used it for every problem on this without realizing it**, like, yeah.

Interviewer: And you used what on every problem on this?

Jackson: Uh, just like **exchanging a different form of what I'm working with if it's more convenient to work with it in that form.**

Because of Jackson's repeated identification of substitution as a strategy for turning a given task into a "more convenient" form, as well as this substitution being allowed as long as the objects being substituted are "equal" within the given context, and Jackson's recognition that these were ideas he used on every task, I claim that Jackson viewed substitution equivalence as a strategy connection across the interview tasks.

Student-generated Connection: Known Equivalences and Own Equivalences

Alex elaborated on a difference between using a *known equivalence* versus defining his *own equivalence* to make substitutions that resulted in the task being easier to complete. I classified Alex as having made a *student-generated connection* because he explicitly discussed how different types of substitution manifested across tasks.

Alex used the metaphor of putting a "mask" on a complicated part of a task in order to temporarily ignore a complicated part and make the task easier to understand. He identified that this "masking" of part of a task was allowed as long as it was temporary, i.e., it is undone at some point before the task is completely answered. On Task 3, Alex applied this metaphor to the process of integration via *u*-substitution. He began the problem by writing the integral $\int_0^{\sqrt{\pi}} 2x\, sin(x^2)\, dx$ to represent the first desired area A, and then used *u*-substitution to evaluate this integral.

Alex: So for that process, that is, um, *u*-substitution, and that is a process of simplifying complex integrals to make them easier to calculate. Um, so the way that works in my brain is I think of it like **taking the difficult parts and putting a mask over them. So they're still there, but you don't have to look at them.** Um, so you can compute it without dealing with the complicated parts of those, **and then once you're done with that, you can take the mask off and put back the parts that you need to.**

Alex then clarified that that in the integral $\int_0^{\sqrt{\pi}} 2x\, sin(x^2)\, dx$, he used u as the "mask" for x^2 and then used his *own equivalence* $u = x^2$ to determine that du should be the "mask" for $2x\, dx$. When asked how the "masked" integral $\int_0^{\pi} sin(u)\, du$ compared to the original integral he wrote, Alex said that the integrands were "not *perfectly* equal because if you were to just say, um, $2x\, sin(x^2)$ is equal to $sin(u)$, that wouldn't be right." However, he then clarified that this substitution was appropriate as long as the substitution $u = x^2$ was made explicit and the final answer for the task was given in terms of the original variable *x* instead of the chosen variable *u*.

Furthermore, Alex expressed that he saw the type of substitution used on Task 3 as a different type of substitution than he had used on other Tasks. For example, when comparing Task 3 to Task 1, Alex said:

Alex: **But that's not a mask that's more switching it,** like the same way as 0.5 and ½ [from Task 1] are equal because that gets you straight to the answer. You don't have to backtrack to get back to the same […] terms that you were in. […] Um, because I can't solve it in terms of *u*. It has to be solved in terms of *x*.

In this excerpt, Alex identified that the substitution of 1/2 for 0.5 in Task 1 would "[get] you straight to the answer," whereas in Task 3, after the substitution of $u = x^2$ was made, he would "have to backtrack" or undo his substitution in order to solve the task in terms of the original variable *x*.

Alex explicitly stated that substitution was a strategy he used across multiple interview tasks, and that some type of equivalence was a necessary condition for these substitutions. In fact, Alex went so far as to say that he used some form of substitution on *every* interview task. However, Alex additionally contrasted problem types by distinguishing between the different ways that equivalence allowed him to use substitution, i.e., using an equivalence that was already known versus one that he introduced himself:

Alex: Um, so while for all of these, **you do need to interchange and you do need to, um, like set them equal to something else to be able to solve the problem**, for *u*-substitution it's necessary to be a mask because you have to be able to take part of that mask off afterwards and be able to plug part of the original part back in. **Whereas for the other ones, you don't have to so it doesn't really need to be a mask, it just needs to be like they're the same in like equivalency, the same in nature.** Um, I think of it as a mask because it still means that there's something behind it instead of being replaced by an equivalent term.

I interpret Alex's use of the phrase "equivalent term" in the last sentence to refer specifically to *known equivalences*. Thus Alex is distinguishing between substitution with his *own equivalences* where there is "something behind" the substitution and he needs to "take […] that mask off" before finishing the task and substitution with *known equivalences* that he can use freely without needing to undo or account for these substitutions later in his problem-solving process.

Discussion

My findings address a gap in the current literature in several ways. First, my work extends the work of previous researchers by shifting the attention from the commonalities that *researchers* identify in students' utterances and work across contexts (e.g., Cook et al., 2022; Richardson et al., 2024) to focus on what *students* explicitly identify as being common across contexts. In particular, Jackson's interview provides evidence that *substitution equivalence* is a commonality that students identified across contexts, and Alex's metaphor of using a "mask" for temporary replacements provides evidence that students identified a distinction between using *known equivalences* or introducing their *own equivalences*. I reiterate that Jackson and Alex were not the only students who explicitly identified these connections, nor were the instances documented here the only connections that students made. Rather, these excerpts provide examples to illustrate that these students made elaborated, conventional connections consistent with what researchers agree is important for students to attend to when their work involves equivalence.

Future research building upon this work could examine the ways in which students' conscious attention to these connections affects their problem-solving. Work can be done exploring the affordances and constraints of students' explicit attention to substitution equivalence and known vs. own equivalences, and how students can leverage their awareness of these ideas in learning new mathematical concepts. As a potential instructional implication, introducing *own equivalences* as a problem-solving method can help student students to turn a complicated problem into one they know how to solve in contexts other than integration by u-substitution (e.g., in Task 4, temporarily replacing $\sqrt[3]{x}$ with a placeholder, or "mask" variable).

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