Students' Generalizing Activity While Using Determinant Applets

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In this article, we share results from a series of interviews with a pair of undergraduates using an applet designed to support students' understanding of determinants. We identify four generalization clusters that the participants developed, and we share our analysis of the activity types that led to these generalizations. We used Ellis et al.'s (2022) Relating, Forming, and Extending framework to characterize the participants' generalizing activity. We share results from this work and discuss the implications for applet design, instruction, and future research.

Keywords: Linear Algebra, Determinants, Generalizing, Dynamic Geometry Software, Inquiry

This paper extends prior work developing inquiry-oriented curricular materials for linear algebra instruction. We designed two web-based dynamic geometry software (DGS) applets intended to provide students with real-time geometric representations of the determinant of a matrix, visualized by the parallelogram/parallelopiped image of the standard basis vectors under a user-determined 2×2 or 3×3 matrix, respectively (Wawro et al., 2023; Mauntel et al., 2024). In the applets, users can manipulate the matrix entries and observe, in real-time, the effect these changes have on the parallelogram/parallelepiped and a dynamic "determinant" value calculated by the applet. Initial implementations of the applets showed strong potential for students' generalizing activity (Wawro et al., 2023). We aim to better understand students' in-the-moment generalizing based on their experiences with the applet and how students' use of the applet might inform design changes to the curricular materials. In service of these goals, we conducted semi-structured interviews with students as they used the applets, prompting them to discuss broader relationships they identified through their activity. We sought to identify (1) what generalizations and generalizing activities do students engage in as they use the determinants applets? and (2) how do these insights into students' generalizing activity inform our applets' design?

Literature and Theoretical Framings

We draw from several research areas to contextualize our work. We first situate our work relative to existing research on student thinking about determinants and dynamic geometry software. We then detail a framework for students' generalizing activity that we use for analysis.

Determinants and Dynamic Geometry Software

In our literature search, we found few studies that attend to student understanding of the determinant concept (e.g., Chagwiza et al., 2021; Ndlovu & Brijlall, 2016). For example, Kazunga and Bansilal (2018) focused on identifying misconceptions and errors used by in-service secondary mathematics teachers. Although most correctly solved determinants of 3×3 matrices, few were successful with 4×4 matrices because many tried to extend or generalize a 3×3 procedure. Some participants also conflated the determinant of a matrix with the matrix itself (evidenced, for instance, by using $|A^T|$ to symbolize the matrix A^T), and many did not use known properties to compute the determinant of a matrix created from one with a known determinant (e.g., the columns were swapped). Through examining pre-service teachers' understanding of determinant using numeric, algebraic, or geometric representations, Durkaya et al. (2011) found that participants employed numeric and algebraic representations of the determinant, but not a geometric one, in answering computational questions; however, during interviews, geometric representations were helpful to improving their understanding.

Some research has focused on integrating technology to support students in developing a conceptual understanding of the determinant. Research shows that Dynamic Geometry Software (DGS) applets allow students to investigate, visualize, make predictions, calculate, simulate, and generalize certain situations, all of which are critical practices for inquiry (e.g., Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014; Greefrath et al., 2018; Hollenbrands, 2007; Paoletti, 2020). In linear algebra, DGS has been used to explore key topics such as linear combinations (Mauntel et al., 2021), linear transformations (Turgut, 2022), eigentheory (Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014), and determinants (Donevska-Todorova, 2012, 2016, 2017). Through applet and task design, Donevska-Todorova (2012) focused on conceptualizing the determinant of a 2×2 matrix as area of the parallelogram defined by the points corresponding to the matrix's row vectors, helping students discover the property $\det(A) = \det(A^T)$ and the relevance of parallelogram point orientation for the determinant's sign. The author attributes the DGS's ability to simultaneously display numeric and geometric feedback as important for connecting to students' prior geometric understanding. Donevska-Todorova (2017) further elaborates on the benefits of DGS in student learning of dot product and determinants, indicating that students took initiative in forming questions and developed a more in-depth conceptual understanding of determinant. Through interaction with the DGS, students indicated evidence of a schematic understanding of determinant by relating it to other mathematical concepts or providing different interpretations of determinants. These studies possibly indicate a potential for DGS as a viable alternative approach to support student understanding of determinant concept, providing them with visualizations of determinant.

In our own design research (Wawro, 2023), we reported on student thinking from interacting with GeoGebra applets designed to leverage students' knowledge of linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ to support an understanding of determinant as a measure of (signed) multiplicative change in the area or volume caused by the transformation. Consistent with the cited literature, we designed the applets so that students contextualize the DGS relative to their existing notions of linear transformations. The goal was to provide students with a foundation for the connections they make while using the applets; in particular, the applets were designed to support students' explorations of how the matrix components [M] relate to the transformed object's graph [G] and to the determinant [D]. In our analysis of students giving observations from the applets and initial explanations for them, we found that most common observation type was [MDG], meaning "[M] implies [D], which makes sense because of [G]." For example, one student wrote, "If any of the columns in the 3×3 are LD [linearly dependent] with another, the $\det(M)$ will be zero due to there being no volume, only area or a line." It was sensible that most student observations began with [M] because the most interactive aspect of the applet was adjusting the matrix component values. This potentially hindered students from conjecturing the biconditionality of the determinant generalizations. Thus, we refined the applets to include two additional features: swapping rows and columns, and dragging the graphed images of the basis vectors that define the parallelepiped or parallelogram.

Generalizations and Generalizing Activity

A core goal of this project is for linear algebra students to extend their reasoning beyond particular empirical examples and develop generalizations that can be used for a broad set of problems and further students' conceptual understanding. The process of generalizing has been highlighted in a number of mathematical areas including algebraic reasoning (Carraher et al., 2008; Ellis, 2007; Kaput, 2008), calculus (Dorko, 2023), computational thinking (Lockwood & De Chenne, 2020; van Borkulo et al., 2021), and linear algebra (Harel, 2017). There are several different theories about how students generalize material. Generalization can begin with students recognizing and continuing a pattern in early algebra (NCTM, 2000). Carraher et al. (2017) highlighted the idea of conjectural generalization, which is "treating an instance (e.g., $7 + 1$) as a case of something more general (e.g., $a + b$)" (p.264). Harel and Soto (2016) explored the idea of structural reasoning, a five-step process that ended with students exploring general structures through a process of pattern recognition and reflection.

To characterize students' generalizing activity, Ellis et al. (2022) developed the Relating-Forming-Extending (RFE) Generalization Framework, which can be used to analyze how students engage in generalizing activity:

> Relating is the identification of similarity across situations, problems, or strategies that the learner perceives as distinct contexts. Forming is the development of an initial, sometimes tentative generalization, and extending is the use of that generalization, sometimes to a broader domain. (Ellis et al., 2022, p. 362)

The relating category involves generalizing based upon a prior context, problem, or situation and can encompass the idea of transfer. The forming category is primary method for creating new generalizations and involves associating objects, finding and isolating consistent features across a number of scenarios, and anticipating or extracting regularities. The extending category encompasses continuing an existing generalization to new situations or cases, making minor or major alterations to the generalization, transforming a generalization through extension, and removing particulars. Technology can aid in the development of generalizations by providing empirical grounding for deductive reasoning and pattern recognition (Turgut, 2022).

Methods

Adopting the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991), the IOLA curricular materials (Wawro et al., 2013) build from experientially real task settings to foster active student engagement in the guided reinvention of mathematics through student and instructor inquiry (Rasmussen & Kwon, 2007). Our research is guided by the Design Research Spiral (Wawro et al., 2022), in which we iteratively draft, implement, and refine task sequences, guided by various design research theories at each phase. The IOLA determinants task sequence (Mauntel et al., 2024) leverages students' knowledge of linear transformations to support an understanding of determinant as a measure of (signed) multiplicative change in the area or volume caused by the transformation. To support students' shift from situated activity to more generalized activity, we leveraged DGS GeoGebra applets that synchronously display dynamic representations of numeric, geometric, and algebraic aspects of the determinant. This makes DGS a suitable tool for supporting students' exploration of determinants, connecting to their existing geometric and algebraic knowledge. DGS applets promote mathematical curiosity (Rumack & Hunker, 2019) as students explore, observe, conjecture, and justify. This centers students' activity as the foundational mathematics to be leveraged and extended. Adopting the RME design heuristic of guided reinvention, the instructor's role includes facilitating students' open-ended activity toward more organized and focused generalizations connecting changes in matrices to changes in the distortion of the space and how they relate to determinants.

Context

We conducted a paired teaching experiment (Steffe & Thompson, 2000) in five interviews with two undergraduates from a small, four-year university in the American Southeast: Ruby and Ahmed (pseudonyms). Authors 1, 3, and 4 identified goals for each interview, which were conducted by Author 1 and assisted by Author 4. In interviews 1 and 2, we oriented the participants to the 2×2 applet and how it depicts aspects of determinants. In interviews 3 and 4, we supported the participants in identifying mathematical relationships through their use of the applets. Toward the end of interview 4 and throughout interview 5, the team focused on eliciting as many generalizations about determinants from the participants as possible.

For this teaching experiment, we used the applets from the initial task sequence (Mauntel et al., 2024, Wawro et al., 2023) and new applets designed for these interviews, which included updated features (Figure 1). First, buttons were added to allow students to switch rows, switch columns, or take the transpose of a matrix (see (a) in Figure 1). Second, randomize and identity matrix buttons (b) were added to facilitate students generating a large assortment of matrices or testing conjectures. Finally, manipulable points were added to the images of the unit square and unit cube (c), allowing students to move the images of the basis vectors that define the parallelepiped or parallelogram (automatically changing the matrix accordingly). The modified applets are found at www.geogebra.org/m/v8hw88fs (2D) and geogebra.org/m/fppgz9q2 (3D).

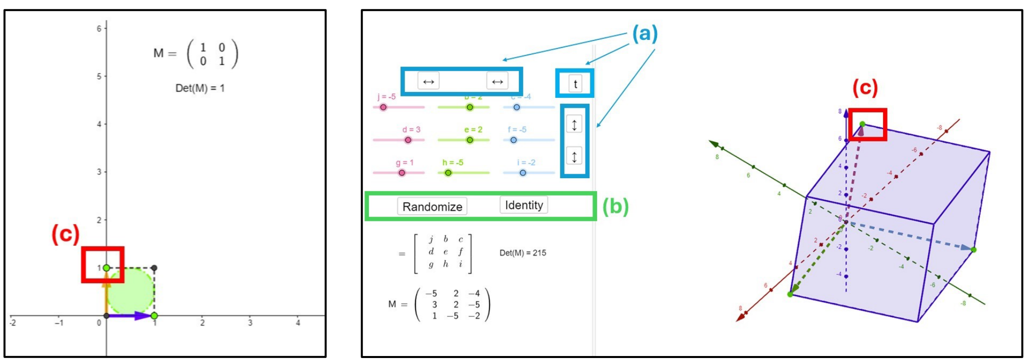

Figure 1. A screenshot highlights the new features of the updated applets.

Analysis

We individually reviewed the transcripts for interviews 2-5 and identified any statements the participants made about determinants that had a conditional or bi-conditional structure. In doing so, we centered the participants' perspectives as they generalized. We holistically categorized these student statements as generalizations through two rounds of coding: open coding and focused coding (Saldana, 2015). The first author combined the identified generalizations onto a single document, merging any instance of repeated identifications. The author team then discussed similarities and differences between the student generalizations. From this, we formed four different generalization clusters (see Table 1) based upon our perceived change to the determinant and used these to categorize participants' activity in later iterations of analysis.

We chunked the transcripts into Episodes of Mathematical Reasoning (EMR) to facilitate the coding process (Andrews-Larson et al., 2024). Each EMR was identified by what we perceived to be the major question the students were trying to answer during that portion of the interview.

The EMRs in the transcripts were primarily separated by a transition between two topics initiated either by the interviewers or the participants' goal-directed activity. For each EMR, we identified which generalization clusters were addressed by the participants' activity.

Using the RFE framework, we coded the participants' generalizing activity according the three broader categories (relating, forming, or extending) and the applicable subtypes of those categories (e.g., within extending: continuing, operating, transforming, or removing particulars). The coding process was divided into three stages. In the initial stage, our goal was to create a shared understanding of the framework's (sub)categories. We then engaged in a second coding phase, during which we noted that codes sometimes repeated across speaking turns due to the nature of the participants' dialogue. Thus, in the third stage, we grouped related chunks together with a single code. Each author coded individually on a cloud-based spreadsheet. During each coding phase, we met regularly to discuss and come to a consensus. We agreed on most codes. In other instances, one or two authors found multiple codes suitable or the interview did not provide enough information for a definitive code; we noted such cases.

Results

Generalizations

Across interviews 2 to 5, we identified four clusters of mathematical activity related to generalizations about determinants (Table 1). These clusters emerged as we sought out specific generalizations and broadened our focus to include the participants' generalizing activity while using the applets. We differentiated clusters by the described effect on the determinant. Cluster A relates to scaling the determinant by k, k^m, or k^n (e.g., if you scale a row by a number k then the determinant is scaled by k). Cluster B includes activity related to negating the determinant (e.g., switching the columns of the matrix). Cluster C mainly consists of characteristics of the matrix that make the determinant zero or nonzero (e.g., vectors are linearly dependent). Cluster D includes descriptions of the determinant remaining unchanged (e.g., swapping rows twice). Although we witnessed Ruby and Ahmed make several distinct, nuanced generalizations that consisted of multiple statements within a given generalization cluster, we do not intend these clusters to indicate that the participants consistently viewed each statement as immediately related to any other within the cluster. Instead, these generalization clusters are an organizational tool used to help identify potentially related generalizing activity across the interviews.

The first two interviews were primarily focused on supporting students' use of the applets and understanding of their connection to determinants and completing the other activities in the unit (Mauntel et al., 2024 (PRIMUS)). The fifth interview was primarily focused on soliciting as many broad, established generalizations from the participants as possible. The excerpts we discuss in this paper are from interviews 2 and 3, which consist of early, initial generalizing activity. Specifically, in this paper, we focus on the participants' generalizing activity that led to Cluster B and explore how this supported generalizing activity toward Cluster A.

Table 1. Clusters of generalizations

Cluster	Effect on Determinant	Mathematical Action/Activity/Statement (Examples)
A	The Determinant is Scaled by k, k^m, or k^n	• A vector is scaled by k • Area/volume is scaled by k. • Multiply a [row/column] by a number k • Multiply m [rows/columns] by a number k

		• Matrix A is scaled by k
B	The Determinant is Negated	• Image is flipped/reflected across an axis • Columns or rows of a matrix are switched • Vectors are swapped (geometric), either instantaneously or dynamically • Vectors cross over each other • Color of the parallelogram/parallelepiped changes • [row/column] is [multiplied by -1 / scaled by -1 / negated]
C	Determinant = 0 or Determinant ≠ 0	• Concepts related to the Invertible Matrix Theorem • Linearly (in)dependent columns/rows • Nonzero volume (3×3) or nonzero area (2×2) • Scaling a row/column by zero/having a [row/column] of zeroes • Rows/columns/vectors are multiples/linear combinations of each other • Cube turns into a plane or a line segment
D	The Determinant Doesn't Change	• Det(I) = 1 • Swapping two rows and swapping two columns • Area/volume stays the same /doesn't change • Determinant stays the same / doesn't change • Changing only an off-diagonal entry • Pushing/leaning the box/cube

The Development of Generalization Cluster B and Cluster A

Reflecting across an axis, scaling rows, and negating the determinant. In Interviews 2 and 3, we asked participants to negate the determinant of given two-by-two matrices. Near the end of Interview 2, Ruby found that the determinant could be negated by reflecting or "flipping" the image of the parallelogram across the *y*-axis. In interview 2, she accomplished this by negating each entry in the top row, saying, "I wanted to flip it across the *y*-axis," later adding, "I changed *a* and *b* [the entries in the top row] to be negative." Ruby also noticed that the color of the parallelogram changed from red to green when she negated the determinant. Around the same time, Ahmed suggested changing the signs of *a* and *c* (the entries of the first column), explaining that doing this causes the result of the 2×2 determinant formula, $ad - bc$, to change from 2 to -2 (in this case). Throughout the episode, we saw several instances in which the participants recognized connections between their actions on two different mathematical objects (here, the geometric image and the entries of the matrix) and changes in the determinant. Consistent with Ellis et al.'s (2022) RFE framework, we coded these as various types of **forming** activity, especially *associating objects*. This is based on their actions on both the geometric (G) and matrix (M) parts of the applet and their observations on how these actions affected the determinant (D). This culminated in a generalization shared by both participants and expressed by Ahmed, saying, "…if you multiply by a negative by the top half of the matrix, you'll get a y reflection. If you multiply by a negative by the bottom half [of] the matrix, you get an x reflection." Throughout these excerpts, the participants consistently scaled rows by -1, which led us to believe that Ahmed intended "multiply by a negative" to mean -1 in this generalization.

Composition of multiple reflections. In Interview 3, the participants reproduced and extended their reasoning from Interview 2, with Ahmed pointing out that the top row determines the *x*-coordinates for the vectors that define the parallelogram, saying, "when you multiply the top row by -1, then you get the flipped image on all flip across stuff *y*-axis." Ruby added, "[to] reflect over x, you change y and to reflect over y, you change x.” Ahmed and Ruby explored this connection by generating examples, which we coded as **extending** by *continuing*. Ahmed then restated their prior generalization, replacing “negative” from before with “negative number.”

> *Ahmed*: …if you multiply the top row by a negative number, you will produce a matrix that is flipped across the *y*-axis, and if, subsequently, if you multiply the bottom row by a negative number, you will produce a matrix that is flipped across the *x*-axis.

In this quote, Ahmed is subtly **extending** his and Ruby’s prior generalization. We note that Ahmed’s statement conflates the matrix itself with the parallelogram image of the unit square under the transformation defined by the matrix. Also, as before, this quote does not mention the determinant. However, Ruby’s activity before and after this quote indicates she was attending to the determinant’s calculation and the color of the parallelogram produced by the applet. Indeed, when pressed by the interviewer a few minutes later, Ahmed stated, “…the determinant would be multiplied by that negative scalar.” Ahmed also clarified that he is applying the negation twice, once to each row, adding, “But if you multiply that negative scalar by both axes, the *x* and *y*, then you'll end up with the same determinant just multiplied by a positive scaling.” Here, Ahmed applied the generalization twice, scaling the top and bottom rows by a negative scalar. This newly formed generalization is structurally different from the former ones because it considers scaling two rows in sequence and anticipating the compounded result. We coded this instance as **extending** by *transforming* because the generalization itself changed for Ahmed.

Scaling by k and extending to k^n. Later in Interview 3, Ahmed asked to update the prior generalization, saying, “So when you're only multiplying the one row by some value, k. k times [row 1], you end up with a determinant times k.” We coded this as **extending** by *transforming,* based on his explicit request to change the language of the prior generalization. Ruby and Ahmed explored this generalization using a specific matrix and scalar to test the statement, which we coded as **extending** by *continuing*. Ahmed then provided an algebraic rationale using a generic matrix with components *a*, *b*, *c*, and *d*; generic scalar *k*, and the 2×2 determinant formula. Like their prior generalizing activity when negating the determinant, Ahmed **extended** this generalization by applying it to each row in a 2×2 matrix, which he stated will cause the determinant to be scaled by k^2 and alluded that this should extend to larger matrices.

Discussion

The most direct result from analyzing Ruby and Ahmed’s generalizing activity has been the four clusters of mathematical activity and statements, providing a foundation for understanding the generalizations that these applets can support. We have also gained insight into how students’ interaction with the applet informs their generalizing activity. Specifically, we recognize that the applet first provides students with examples of matrices to explore during their **forming** activity. Students also often return to the applets during **extending** by *continuing* activity to explore more specific cases to check established but untested generalizations. This suggests potential design changes. For instance, we could incorporate features that can be toggled on once students have developed a specific generalization so that the applet might support students’ **extending** activity. Though space limitations prevented us from exploring it much here, we have found some evidence that the updates to the applet supported some of the generalizing activity.

Another contribution of this work is the application of Ellis et al.’s (2022) framework in a new content area and with an additional focus on generalizing with DGS applets. Most notably, we have identified a clear instance of **extending** by *transforming*, which Ellis et al. state “are not well specified in the literature” (p. 377). This demonstrates a meaningful contribution to the field’s understanding of undergraduates’ generalizing activity.

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