Learning Linear Algebra Through Instrumentation of Interactive Figures: Analysis of Student-Generated Mathematical Observations and Conjectures

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Linear algebra is an important topic with many in-demand applications. In this paper, we consider a set of digital interactive figures created for introductory linear algebra students. The figures were designed to scaffold student experimentation with linear algebraic objects in the process of making observations and conjectures. We analyzed a collection of student reflections written about their experience and observations from the interactive figures through the theory of instrumental genesis. Based on the results of this pilot study, we believe that these interactive figures can be effective in promoting cycles of observation and conjecture as pre-proof activity.

Keywords: Linear algebra, CAS, eigenvalues, conjecture, instrumental genesis

Literature Review

Linear algebra has been and continues to be a crucial foundational topic for many other disciplines. More recently, the modern era of data-driven and machine learning innovation has only increased its relevance. As a deeply theoretical yet also readily applied subject, the way that linear algebra is taught is of critical importance as it needs to lead students to conceptualize and yet think pragmatically.

Since the early 1990's the Linear Algebra Curriculum Study Group (LACSG), and contemporarily the LACSG 2.0 in 2022, have set forth an educational agenda for the teaching of linear algebra in the university mathematics curriculum (Carlson et al., 1993; Stewart et al., 2022). Among this guidance, of primary relevance to this project are the recommendations to implement and integrate computer algebra systems (CAS) learning support, and to further attend to the theoretically based content common to second courses in linear algebra.

In keeping with this movement, modern linear algebraic texts often encourage and integrate the use of CAS support such as MATLAB or Maple (Lay et al., 2016; Leon, 2010) Using CAS support to assist students in performing computations involving vector and matrix manipulations, solving systems of equations or computing matrix determinants, inverses, and eigenvalues and eigenvectors allows for fast and accurate calculations. The intent is to leverage CAS to reduce a student's cognitive load on routine arithmetic, thus computing should provide an arena for experimentation and tinkering that can help reinforce conceptual ideas, such as by "generating computational justifications for claims and making strategic choices to limit [complexity]" (Bagley & Rabin, 2016). This experimental mode of doing mathematics via computation also possesses affective consequences, as students may be able to shed restrictive, formulaic views of what "doing" mathematics is (Castle, 2023). Conversely, using CAS may distract students from focusing on linear algebra and shift their attention to the coding itself. As such, in implementing CAS support in the teaching of linear algebra, it is important to understand what impact it is having on student learning.

In order to connect ideas of computation and experimentation with productive modes of mathematical activity, we utilize an emergent model for the process by which mathematicians engage with mathematics, Observation-Conjecture-Proof-Theorem (Peffer, McDonald, & Stewart, 2024). Rooted in traditional Definition-Lemma-Proof-Theorem-Proof-Corollary

(DLPTPC) models, which can be opaque and intractable to introductory mathematics students (Uhlig, 2002), the Observation-Conjecture-Proof-Theorem model of doing mathematics seeks to delineate a simple path by which students can generate, critique, and validate mathematical ideas. In an introductory linear algebra course, we further wish to primarily engage students in routines of observation leading to conjecture, which can be motivated and facilitated by CAS-automated algebra and computation.

Theoretical Perspectives

In this study, we will use Trouche's (2020) model of instrumental genesis to understand how students come to understand and utilize technology (interactive figures) to perform a task. The theory of instrumental genesis describes a codependence between a subject (prototypically the student learner) and the artifact (e.g. a calculator, graphing tool, or digital intractable).

In quantifying the artifact, the affordances/potentialities and constraints of the artifact become the relevant leverages by which the subject and artifact interact. The affordances/potentialities of an artifact are "what it offers [the student], what it provides or furnishes," and the possibilities for mathematical activity that are opened to the student. In contrast, the constraints of an artifact are those limits that bound the space of action and perception available to the subject.

Thereafter, the theory of instrumental genesis describes twin processes of instrumentation and instrumentalization, which describe the mutual changes elicited in the subject by the artifact, and vice versa upon the artifact by the subject. Instrumentation is "the action by which a subject acquires an artifact, and the effect of this action on the subject, who develops, from this artifact, an instrument for performing a task. An instrument is thus made up of an artifact component and a cognitive component (knowledge necessary for/from using the artifact for performing this type of task)" (Trouche, 2020). Conversely, the process of instrumentalization describes how a subject adapts an artifact to their intentions, via stages of discovery (of functions), personalization, and customization.

Within this study, the artifact(s) in question are a collection of interactive figures developed by the second author, meant to host a space for exploratory linear algebraic arithmetic action and observation. These interactive figures primarily feature concrete, number-valued matrices and vectors, and different calculations are performed by the figures. The affordances of the figures include the mechanisms for control at the student's disposal, largely the "new matrices," "size of matrices," or scalar setter buttons. In addition, the automatic display of the example matrices generated and their relevant sums, products, scalings, and exponentiations, as well as their corresponding calculated eigenvalues constitute intended structural constraints of the figures.

As identified by the interactive figures' constraints, the interactive figures leave very little room for personalization as the source code is unavailable to students, and the discovery process is intended to be near immediate to a population of students who are familiar with mechanisms such as buttons and sliders. As such, we will not explore the instrumentalization process and instead primarily emphasize the artifact level affordances and constraints as well as the instrumentation process, by which a subject's understandings and strategies evolve as a result of working with the artifact.

Thus, we seek to utilize the framework of instrumental genesis, in conjunction with our Observation, Conjecture, Proof, Theorem conceptual framework of doing mathematics to investigate the in-situ affordances/potentialities and constraints experienced by the students, and what observable effects upon the student via the instrumentation process can be observed.

Our research questions are:

- How did the interactive figures facilitate (or not) students to participate in the Observation-Conjecture-Proof-Theorem model of doing mathematics?
- What was the process of instrumentation of interactive figures for linear algebra students, and what were the experienced affordances and constraints of the interactive figures?

Method

This case study was conducted in an honors section of a first course in linear algebra at a large public land-grant R1 university in the northwest USA. Out of 16 enrolled students, 13 participated, with most of those students interested in quantitative degrees in STEM fields and medicine.

With the intention of developing a conceptual framework by which to evaluate the interactive figures' merits, such as by collecting student perspectives on the interactive figures' instrumental affordances and constraints, a single digital worksheet activity was assembled consisting of five interactive figures related to the algebraic properties of eigenvalues (See Figures 1-4 below). As intended introductory or pre-introductory materials, the interactive figure activity was administered to students over a weekend's time to complete following a brief first lecture on eigentheory. As such, students had been shown the definitional equation of eigenvalues and eigenvectors of a matrix as well as basic necessary conditions such as non-zero eigenvectors and uniqueness of eigenvalues for an eigenvector. By the time of participation, students had not seen nor practiced how to compute eigenvalues or eigenvectors, nor had any of the eigenvalue properties displayed in the worksheet been discussed in class.

The student participants were assigned to spend approximately 15 minutes manipulating the digital figures and look for any observable patterns or trends. For course extra credit, the students then submitted one to three paragraph reflections on their observations made based on the following open-ended prompt: Write out your thoughts in a few paragraphs about whatever you found interesting or noteworthy while in the worksheet. These reflections were gathered via the course space in Canvas, which were collected and de-identified in a OneDrive file.

From these reflections, the authors used open coding (Strauss & Corbin, 1998) to establish themes and build a codebook, which include the following main codes: Descriptive language, Eigenvalue, Eigenvector, "Why" Questions, Worksheet (Figures, Features, Design), Affect, Mental Acts, Agreement/Believe as Fact, and Examples. Furthermore, we have extracted all mathematical statements included in the student reflections to analyze the stage of mathematical activity present in the Observation-Conjecture- Proof-Theorem model of "doing" mathematics.

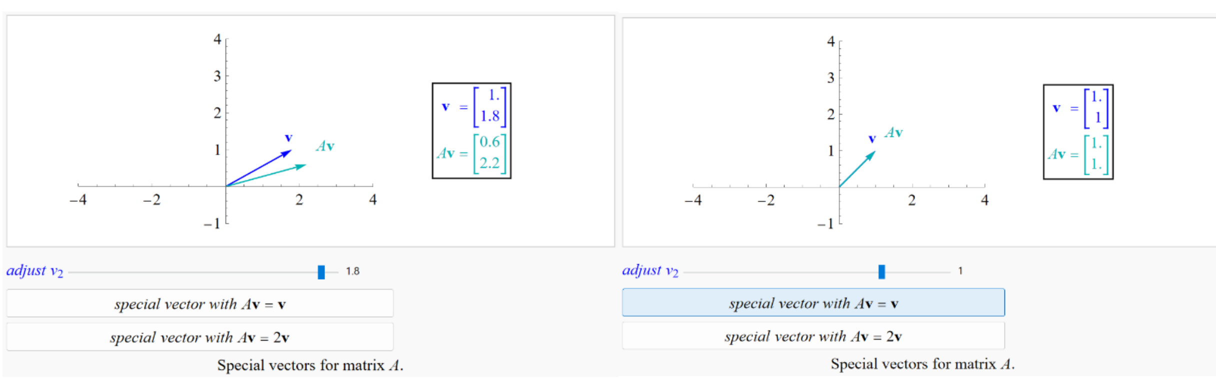

Figure 1. Geometric and algebraic connection of eigenvalues and eigenvectors.

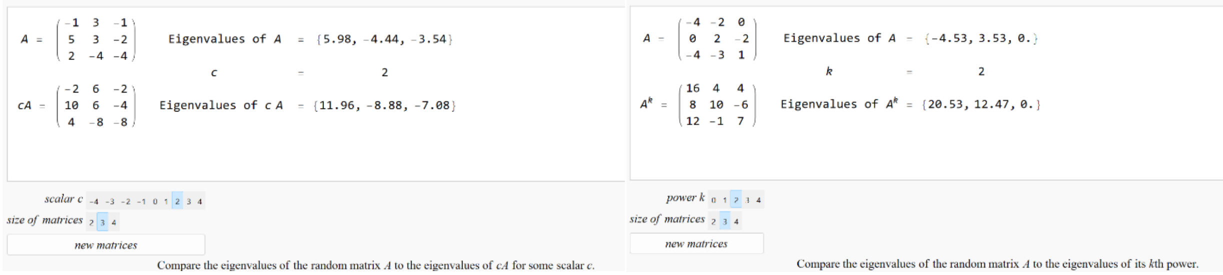

Figure 2. Scalar multiplication and matrix exponentiation on eigenvalues.

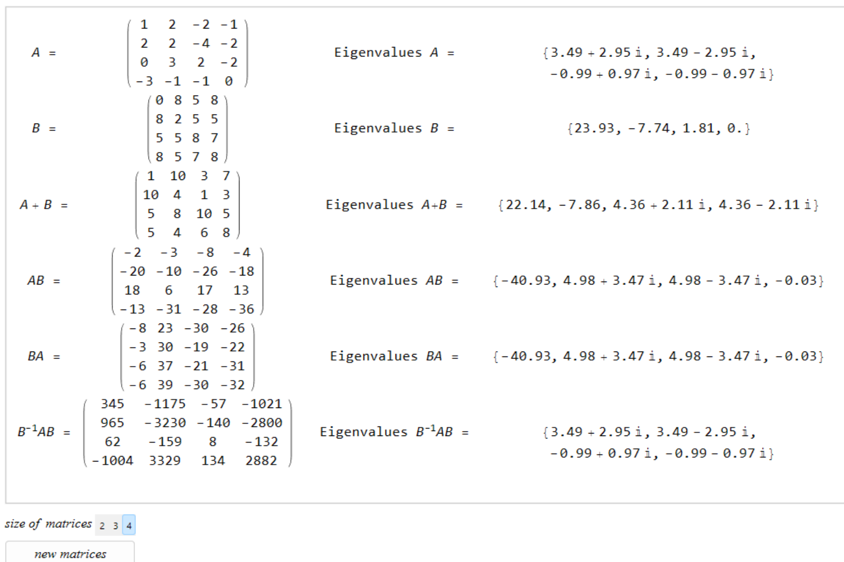

Figure 3. Eigenvalues and sums and products of matrices.

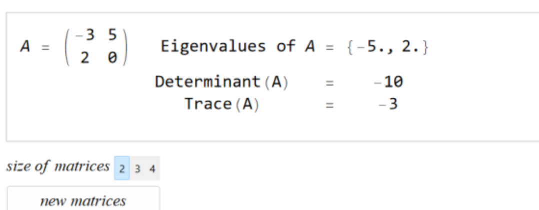

Figure 4. Relationship between eigenvalues, determinant, and trace.

Results and Discussion

Students' Construction of Mathematical Statements

Of the 13 participants in this study, 12 students attempted to discuss in some capacity the algebraic relationships given in the interactive figures. Of these students, 7 produced most or all of the desired relationships (or non-relationships, in regard to sums and products of matrices), with three others largely stating that they did observe patterns without explicating them and the final two stating that they did not understand how to interpret the figures and could not construct any of the relationships.

Scalar multiple and matrix exponentiation. Each of the 7 identified participants were able to construct valid relational statements about the eigenvalues of matrices cA or A^k from the interactive figures shown above in Figure 2, such as the excerpts below. We noticed a variety of levels of generalization demonstrated by students, such as S1 who states the property for scalar multiples for any scalar c, while S2 only constructed the relationship for c = 2. Whether or not that student understood that the property extended to any other constant and used 2 as a prototype is unclear. Furthermore, we can observe that even though the worksheet only allowed students to select scalar values from integers between -4 and 4, students such as S1 were comfortable extending this observation to include any scalar.

> S1: "Multiplying a matrix by a scalar scales the eigenvalues by that same value"
> S2: "The eigenvalues of 2A are twice the amount as the eigenvalues of A."
> S5: "When the matrices were put to a specific power, the eigenvalues were also put to that same power."

Sum and products of matrices. Consider the more open-ended figure relating the eigenvalues of sums and products of matrices (Figure 3 above). Seven of the participants commented in some capacity that there is no identifiable pattern from the eigenvalues of A and B to the eigenvalues of the sum A+B or product AB. Nebulously, two students reflected that they could see the pattern between sums and products of matrices without explicating what pattern they were or were not seeing.

Of those students who correctly observed a lack of relationship in the matrix sum and product example, we notice a distinct difference in the style and conviction of the statements themselves. For example, S3 phrases their statement definitively whereas S5 couched their language in more uncertain terms, in particular leaving room for the possibility that they just couldn't find the intended relationship.

However, this uncertainty seemed to only be present in the case of non-patterns that the students might have expected to find, as all 7 conjecturing students presented definitive statements about the relationship between the eigenvalues of AB and BA or $B^{-1}AB$ and A, as with the final two excerpts from S4 and S6 below.

> S3: "When you add two matrices together, the eigenvalues do not correspond to the addition of the eigenvalues."
> S5: "I could not find the relationship between eigenvalues when the matrices they stemmed from were added or multiplied to each other."
> S6: "Multiplying two matrices together [does not necessarily] multiply the eigenvalues. The resulting eigenvalues always seem to be slightly greater in magnitude than the result one would anticipate."
> S4: "If a matrix A is multiplied by a matrix B, it's eigenvalues will be the same as Matrix B times Matrix A. Also $B^{-1}AB$ has the same eigenvalues as just A."

Extent of observation-conjecture-proof-theorem activity. We have seen evidence of students effectively engaging in observation and conjecture style mathematical activity through participation with the interactive figures. The below two excerpts thus demonstrate where students have stopped in the process of Observation-Conjecture- Proof-Theorem.

> S1: "As to the specific eigenvalues of matrixes, we haven't discussed how to find them, so I'm taking the worksheet's values as fact."

S3: “I also liked the brief explanations that they gave, however, I would like to have some more in-depth discussion about how to solve eigenvalues.

As expected, the students were not able to leverage the interactive figures in order to glean further conceptual meaning or computational technique about eigenvalues and eigenvectors. In this way, the domains of verification and justification as proof-type activity are not currently accessible to students following their time with the interactive figures, however by their remarks we believe that the interactive figures have helped to prime students for these further deeper conversations.

Artifacts of Instrumentation: Affordances and Constraints

The below three excerpts each contain some language relating what the student was able to “do” with the interactive figures. For S2, changing the size of the matrix being exponentiated helped to identify the relationship, at least to them. S3 remarked about the ability to change matrices, which likely came from the ability to either modify the matrix dimension or change parameters such as the scalar multiple or exponent. Finally, S9 used embodied language “scale the vector with my hand” when referring to the geometric interactive figure, indicating a level of experienced control. All of these comments indicate that the designed features of the interactive figures were largely experienced as affordances of the figures that students were able to leverage to help make sense of the encoded relationships.

S2: “For the powering up a matrix … When I made it a 3x3 matrix, it looked like the the eigenvector/the matrix of A^k is the product of A times A, and the eigenvalues of A^k = the eigenvalues of A to the power of k, but I don't know why some of the signs changed from positive to negative if it was being squared.”
S3: “I found the worksheet to be useful in that it allows you to change the matrices and see what happens to the eigenvalues.”
S9: “It was interesting to scale the vector with my hand and check the gradual change of the values on my own.”
S8: “For number three [Figure 3], I also get the products of eigenvalues and the different ways you can add or multiply them together, but like before I do not get how you created the eigenvalues in the first place.”

Conversely, the only constraint that was remarked on by students was that the operations performed to calculate eigenvalues were concealed, for example the above excerpts from S8 which communicate that the student felt limited in their ability to engage with eigenvalues and eigenvectors as they have no ability to compute and manipulate eigenvalues outside of the figures.

However, no student remarked upon other elements of the tight boundedness of the figures’ functionality such as not being able to edit matrices entry-wise, perform calculations other than those given, or change dimensions or parameters beyond the set range given. While these constraints were nonetheless present, it is unclear whether the students did feel limited by this structure and did not remark upon it, or they were sufficiently engaged by the afforded freedoms that those constraints did not impede their investigation. Further work investigating students’ thoughts while using the interactive figures could help make this distinction.

Trouche (2020) defined instrumentation as “The action by which a subject acquires an artifact, and the effect of this action on the subject, who develops, from this artifact, an instrument for performing a task.” (p. 407).

From the above student excerpts, S3’s statement indicates that the student felt they were in control of manipulating the matrices, even though none of the interactive figures allowed for a direct entry-wise manipulation of the matrices, while S9’s comment uses embodied language of performing actions by hand. Both of these we interpret as the students actively manipulating the interactive figures in order to pursue a goal, in this case to make observations of the algebraic and geometric objects, and thus the students have, to some degree, instrumentalized the interactive figures for the purpose of engaging in Observation-Conjecture-Proof-Theorem activity.

Concluding Remarks

This study sought to investigate the types of activity in an Observation-Conjecture- Proof-Theorem perspective of doing mathematics that students engaged in while working with the interactive figures. We saw that students were able to effectively make reasonable and coherent mathematical statements relating algebraic properties of eigenvalues based upon their observations in the interactive figures, and that we believe this activity to have helped prime the students for further conceptual and computational development of eigen-theory. Additionally, the theoretical framework of instrumental genesis was useful and operationalized to examine how students experienced and engaged with the interactive figures in the action of making observations and conjectures.

In conjunction with these interactive figures focused entirely on properties of eigenvalues, other figures have been made that also display eigenvector information. Furthermore, the second author has designed several other sets of interactive figures on the topics of linear combination and span, linear transformations, matrix multiplication, and determinants, to be used across introductory and second courses in linear algebra.

Since these interactive figures are solely intended to scaffold and elicit observation and the potential for conjecturing activity, the first and second authors are in the process of integrating these interactive figures more intentionally into an introductory linear algebra curriculum by supplementing them with explicit conjecturing and proving assignments. From this pilot investigation, we believe that students are able to instrumentalize these interactive figures for the task of investigating algebraic relationships, and thus we hope that students can further utilize those observations and conjectures in the mathematical process of Observation-Conjecture-Proof-Theorem by subsequently supporting their conjectures via proof.

Furthermore, we were not able to follow up with students to make any assessment of retention or long-term effects of this interactive figure worksheet in their learning of linear algebra. Furthermore, the reflection prompt was left purposefully open-ended so that students could direct their writing towards whatever they found notable about the worksheet, meaning only 7 of the 13 participants attempted to quantify the algebraic relationships present. In future iterations of this research, we plan to pair investigations of these interactive figures in introductory linear algebra courses with subsequent proof activities that structure and draw from the observational affordances of the interactive figures. In doing so, we wish to better understand how these interactive figures can be operationalized and instrumentalized by students to accomplish a task, in this case observing and conjecturing quantifiable algebraic relationships in order to further prove them.

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