An Exploration of Productive Student Reasoning About Equivalence Relations

Hudson Payne
Oklahoma State University

John Paul Cook
Oklahoma State University

Equivalence relations are a foundational concept in mathematics, yet productive student reasoning about equivalence relations remains largely understudied. Based on the descriptions of productive reasoning about equivalence that we have encountered in the literature, and the results of the data we collected from task-based interviews with advanced mathematics students, we argue that an essential aspect of productive reasoning with equivalence relations is the flexibility to transition between an element-wise, local perspective of the equivalence relation and an equivalence class level, global perspective of the equivalence relation.

Keywords: Equivalence Relations, Equivalence, Local Perspective, Global Perspective

Introduction

Equivalence relations are one of the most foundational concepts across the entire mathematics curriculum, ranging from numerical equality in early childhood (e.g., McNeil, 2014) to abstract algebra at the advanced undergraduate level (e.g., Cook et al., 2022). Additionally, as a prominent feature of many introduction-to-proof courses, various textbook authors describe equivalence relations as "a surprisingly powerful tool" (Dumas & McCarthy, 2007, p.59) "particularly valuable" (Smith et al., 1986, p. 145), "quite significant" (Hammack, 2013, p.184), and "extremely important" (Roman, 2006, p. 141). Yet despite their centrality in mathematics, student reasoning about equivalence relations, and in particular *productive* student reasoning about equivalence relations, remains largely understudied. In this report, we will synthesize the literature on students' reasoning about equivalence relations and use task-based interviews with an advanced mathematics student to provide an answer the question: What does productive reasoning about equivalence relations entail?

Based on the descriptions of productive reasoning about equivalence that we have encountered in the literature, and the results of the data we collected from the interviews, we will argue that an essential aspect of productive reasoning about equivalence relations entails a flexible transition between an element-wise local perspective and an equivalence class-based global perspective. These ideas are implicitly present in the existing literature about equivalence relations, but have not yet been explicitly articulated or empirically substantiated. The articulation and demonstration of reasoning from both local and global perspectives and the benefits of using these perspectives flexibly will be the primary contributions of this paper.

Literature Review and Theoretical Perspectives

This study is part of a broader effort to develop and refine a *conceptual analysis* (Thompson, 2008) that describes and accounts for students' reasoning about equivalence across mathematical domains. In alignment with the tenets of conceptual analysis, our efforts here aim to identify and describe elements of students' reasoning "that might be propitious for [their] mathematical learning" (Thompson, 2008, p. 60). The literature on students' reasoning about equivalence, broadly defined, is vast. For example, there is a wealth of research focused on how students productively reason about specific instances of equivalence in K-14 settings (e.g., Cook, Dawkins, & Reed, 2021; Jones & Pratt, 2012; Kieran, 1981; Smith, 1995). Here, however, we aim to examine how advanced mathematics students might productively reason about tasks that involve an unfamiliar, formally defined equivalence relation. As suggested by quotations in our introductory paragraph above, these kinds of tasks are typical in proof-based mathematics courses. Indeed, while some of the challenges and difficulties of reasoning about such tasks have been reported in the literature (e.g., Brandt, 2013; Chin & Tall, 2001), descriptions and empirical examples of what constitutes productive reasoning has received substantially less attention. This is the primary gap we aim to address in this study.

The literature does provide some theoretically-based hypotheses about what productive reasoning about equivalence relations might entail, but they have not yet been empirically examined with students. Our primary argument in this paper (and the answer we offer to our research question) involves empirically examining one of these theoretical hypotheses: a key element of productive reasoning about equivalence relations entails viewing the relation through (and moving flexibly between) both a *local perspective* and a *global perspective* on the relation. A *local perspective* involves reasoning about whether or not specific pairs of elements are related with respect to a given equivalence relation. Researchers, however, have hypothesized that productive reasoning about equivalence relations requires going beyond this familiar element-to-element perspective to consider classes of related elements. Hamdan (2006), for example, argued that students' should be supported in "envisioning the equivalence classes corresponding to an equivalence relation, which requires going beyond the element-wise conception of a relation, to a more *global* perception of it" (p. 130-131, emphasis added). Similarly, Schumacher (1996) argued that moving flexibly between the relation (to focus on relationships between specific elements) and classes of related elements (the partition structure induced by the relation) "are two faces of the same problem. […] These two views complement each other, and it is useful to be able to cross over easily from one to the other" (p. 46). In both instances, we infer that the authors are acknowledging the importance of a local perspective on the relation but are also arguing that a focus on equivalence classes (or, more broadly, collections of related elements) is critical – other researchers have advanced similar theoretical arguments (e.g., Chin & Tall; Asghari; Asghari & Tall (2005)Therefore, following Hamdan's (2006) use of the term, we characterize a *global perspective* of an equivalence relation to involve reasoning that focuses on collections of related elements. We hypothesize that utilizing both perspectives is a key aspect of productive reasoning with equivalence relation tasks.

Methods

We employed the task-based clinical interview methodology (e.g., Goldin, 2000; Clement, 2000) in order to explore students' organic reasoning processes. Specifically, we conducted individual interviews with four graduate students studying mathematics at a large R1 university. Each student participated in two interview sessions conducted by the first author; each session was up to 2 hours in length. Since this project was largely exploratory in nature, we chose to work with students who would already have strong conceptions about or extensive experience with formal equivalence relations in order to observe and draw inferences about the nature of productive reasoning. In this short report, we focus on our analysis of the activity of one of the four students (pseudonym: Aspen). All interviews were recorded using an iPad application, which produced videos of students' written work with synchronized audio.

The interviews centered on a series of literature-informed tasks, each of which were derived from the exercises involving equivalence relations in proof-based mathematics textbooks. Per our research hypothesis, the tasks were designed to (a) be typical in form yet unfamiliar enough to require legitimate mathematical reasoning to solve, (b) be open-ended (to encourage multiple approaches), and (c) elicit reasoning from both a local perspective and a global perspective. The task we focus on in this report is featured in Table 1.

To analyze the data, we conducted a theoretical thematic analysis (Braun & Clarke, 2006) that was organized around our characterizations of *local perspective* and *global perspective.* The interviews were transcribed, and the first author reviewed the transcripts, first classifying the approaches to each of the tasks as either productive (successful) or unproductive (unsuccessful), and then coding using notions of *Local Perspective*, *Global Perspective*, and *Transition*. Additional codes were applied to further describe students' approaches to each task. Throughout this process, the first author chose excerpts from the transcripts to discuss and review with the second author. These excerpts and the application and interpretation of the codes was discussed and refined, at which point the first author would revisit the data in light of any revisions or new insights that occurred.

Table 1. Statement of and rationale for one of the interview tasks.

Task:	Rationale:
Let $(x, y)R(w, z) \Leftrightarrow x^2 + y^2 = w^2 + z^2$ be a relation that partitions the plane into curves. What is the maximum distance between two points on curves containing (1,2) and (0, 4)?	This is a modified version of a task from Smith, Eggen, and St. Andre's (2006) introduction to proof text. We included it because of its potential to elicit the use of a local perspective, a global perspective, or both.

Results

Our analysis and presentation of the results here focuses on the responses of one participant (Aspen) to the task in Table 1. We first discuss what we interpreted as Aspen's productive use of a local perspective on the equivalence relation and then do the same for her productive use of a global perspective.

Demonstrating Productive Use of a Local Perspective on the Equivalence Relation

Aspen began by checking to see whether (0,4) and (1,2) (the two ordered pairs given in the statement of the task) were related. She concluded that they are not by using the condition for the relation – see Figure 1 (left).

Figure 1. Aspen reasons that (1,2) and (0,4) are not related (left) and sketches various points related to (1,2) (in green) and (0,4) (in yellow), respectively (right).

Aspen then generated points that were related to each of these two points. She described her approach by directly referencing the statement of the equivalence relation and noting that, because $0^2 + 4^2 = 16$, the points related to (0,4) "have to be the x coordinate squared plus the y coordinate squared being 16." (She reached a similar conclusion about the points related to (1,2) and the sums of the squares of the coordinates as 5.) She proceeded algebraically by attempting to 'guess and check' some tractable values for x and y that might satisfy this condition. The elements she identified as related to (0,4) included ($\pm$4,0), $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$, and $\left(\pm 2\sqrt{2}, \mp 2\sqrt{2}\right)$. Similarly, for (1,2), she identified $\left(0, \pm\sqrt{5}\right)$, $\left(\pm\sqrt{5}, 0\right)$, $(\pm 1, \pm 2)$, and $(\pm 1, \mp 2)$. We interpret that Aspen was demonstrating a local perspective here because she was focusing on relationships between pairs of points – specifically, (0,4) and whichever new point she had generated. This approach helped her develop familiarity with the equivalence relation and paved the way for her to turn her attention to collections of related elements, which we detail in the next section.

Demonstrating Productive Use of a Global Perspective on the Equivalence Relation

After generating these points by demonstrating what we interpreted as a local perspective, Aspen concluded that "it's not gonna be a ton of nice points," and then explained that she was "just gonna plot some points on this curve, see if I get anything that resembles a pattern." In her rough sketch, she used yellow to denote the points related to (0,4) and green to denote the points related to (1,2) – see Figure 1 (right). After she had sketched these points, she observed that "I think it's gonna be like a circle." When asked why, she explained:

> *Aspen*: I think this because I did 4 points on (0, 4) in the yellow one, um, and it happened to do that [referring to the four points on the yellow circle that intersect the axes]. […] This is $\left(2\sqrt{2}, 2\sqrt{2}\right)$, which also would work if it was $(-2\sqrt{2}, -2\sqrt{2})$. Same thing if one of them was negative more than the positive. […] If this, say, is my point that is $(2\sqrt{2}, \pm 2\sqrt{2})$, um, then I'll have the same exact point over here, over here, and over here, approximately [referring to the points of intersection of the circle and the lines $y = \pm x$]. […] So I think they're both gonna be circles.

Aspen's realization that the curves containing $(1,2)$ and $(0,4)$ were both circles was particularly propitious for her progress on this task, as she was then able to reason that the maximum distance between points related to (1,2) and points related to (0,4) could be quantified by reasoning about the radii of the two circles:

> *Aspen:* So really, if they're both circles, then the distance is gonna be the same at all the points. [...] I think *that* [referring to the length of the red line segment along the x-axis she drew from (-4, 0) to ($\sqrt{5}$, 0) - see Figure 2] would be the maximum distance. Because if I took any point on, say, the edge and took it to the opposite edge, that should give me [...] $2\sqrt{5}$ because that's the diameter of the [green] circle […], plus, uh, $4 - \sqrt{5}$.

In this way, Aspen (correctly) reasoned that the length of the red line segment could be computed by adding the length of the diameter of the green circle ($2\sqrt{5}$) to the difference between the lengths of the diameters of the two circles ($4 - \sqrt{5}$).

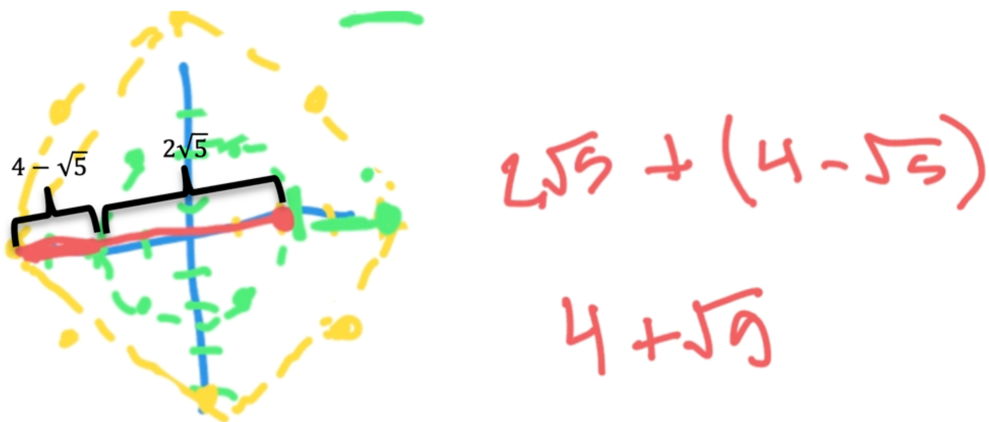

Figure 2. Aspen's red line segment, the length of which she reasoned was the desired maximum distance (we have labeled the two lengths she calculated), and her calculation of its total length.

In a subsequent interview session, the interviewer proposed two alternative line segments whose lengths, he suggested, might also represent the maximum distance (see Figure 3).

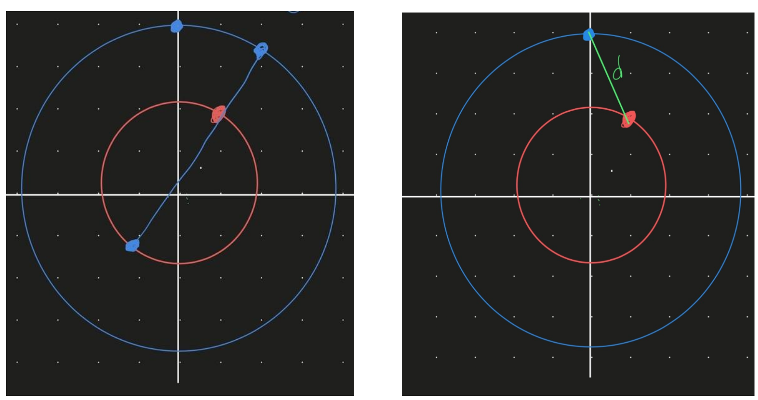

Figure 3. Interviewer's first (left) and second (right) alternative responses to the task.

After the interviewer prompted her to consider the image in Figure 3, left, Aspen responded that she also considered this a legitimate solution:

> *Aspen:* I would agree that this is the... It's the same concept that I have drawn here [referring to her own approach in Figure 2]. It's just using a different angle, but it's still gonna be the maximum distance. [...] The radius of the circle is always gonna be the same no matter where you're picking your point, so we are always gonna have this diameter of this circle in red, and then the difference between the, uh, [lengths of the diameters of the] blue circle and the red circle in this picture. [...] There are easier points to pick. I would say that this one is not the easiest point, um, but it doesn't matter which point you pick.

Aspen disagreed, however, that the second approach in Figure 3 was correct, arguing that the proposed length was not the maximum distance. She explained, however, how one could attain the maximum distance by 'moving' the red point along the red circle:

> *Aspen*: I would say that there can be a further distance. And, how I'd respond to that is say

your blue point on the blue [outer] circle is fixed. You can technically move this red point on the red [inner] circle as long as it stays on the red circle. [...] I can move this point anywhere on the circle […]. Is the distance getting bigger or smaller? And then, if it's getting bigger, then where's gonna be the biggest possible? And, hopefully, they would say that should be this point here [gesturing to $(0, \sqrt{5})$]. [...] You can't move in the between space.

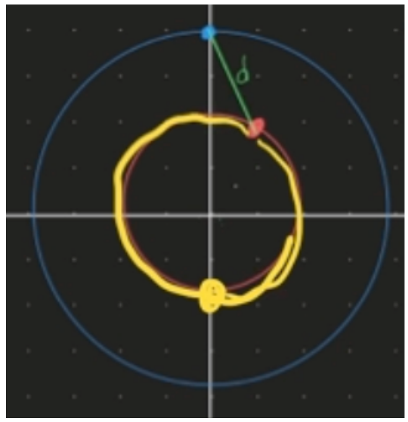

Figure 4. Aspen explains how to modify a proposed solution to attain the maximum distance.

We interpret that Aspen, after starting her response to this task by reasoning with a local perspective, had shifted to a global perspective because she began reasoning about collections of related elements. As evidence, consider that, after plotting specific instances of related points (that she had generated using a local perspective), she asserted that "they're both gonna be circles." Additionally, all of her subsequent comments about the maximum distance referred to these circles and specific distances between them (and not to specific pairs of related elements). Though Aspen identified specific locations on these circles (as in Figure 4), she did not specify coordinates for them, even arguing that "it doesn't matter which point you pick." This shift from local to global also involved a shift from algebraic reasoning (in which she used a 'guess and check' approach to identify pairs of values of x and y that would satisfy $x^2 + y^2 = 16$ or $x^2 + y^2 = 5$) to geometric reasoning (in which she plotted points in the Cartesian plane to observe graphical properties of the curves containing these points).

Discussion

In this report, we posed a question that has received very little direct attention in the literature: what does productive reasoning about equivalence relations entail? To answer it, we empirically examined and elaborated a theoretical hypothesis from the literature: that productive reasoning about equivalence relations entails attention to specific related elements as well as classes of related elements. We characterized the former as a *local perspective* and the latter as a *global perspective* on an equivalence relation and used these constructs to analyze the reasoning of an advanced mathematics student in task-based clinical interviews. We believe our analysis here supports and provides empirical substantiation for the literature-based hypothesis that, with regards to a local and a global perspective, "it is useful to be able to cross from one [perspective] to the other" (Schumacher, 1996, p. 46).

Use of both a local perspective and a global perspective supported Aspen's completion of these tasks in concrete ways. Using a local perspective, for example, enabled Aspen to develop familiarity with the relation and generate a list of pairwise equivalent points via algebraic techniques. Generating these points was critical because it enabled her to identify the geometric patterns exhibited by the equivalence classes and thus facilitated the shift to a global perspective. Using a global perspective enabled Aspen to reason geometrically about the situation – specifically, she was able to use geometric properties of circles to efficiently identify that a way to compute the desired maximum distance without referencing any particular coordinates of points on those circles. This was a notable departure from her local approach, which centered on particular points and their coordinates.

Our primary contribution, therefore, is the description, analysis, and illustration of two perspectives that can support students in reasoning about equivalence relations. Through our conceptual analysis lens, in other words, we consider the local and global perspectives to "be propitious for students' mathematical learning" (Thompson, 2008, p. 60). This contribution addresses two gaps in the equivalence literature. First, while the equivalence literature is substantial, few studies have investigated students' reasoning with formal equivalence relations, and those that have are largely focused on the difficulties students encounter. This study contributes to the literature by provided theoretical descriptions of what students' *productive* reasoning could or should entail. Second, very little research had previously examined students' reasoning about equivalence from a global perspective, perhaps because all of K-14 mathematics approaches equivalence from a local perspective. This study therefore contributes to the literature by analyzing a student's use of a global perspective and how it supported her successful completion of a somewhat typical equivalence relations task; it also offers some insights into how a global perspective might be productively used in conjunction with a local perspective.

We acknowledge that our characterizations of a local perspective and global perspective – though useful classifications of students' reasoning – are very broad. They do not, for example, afford insight into *how* and *in what ways* students reason locally or globally. And while the literature does provide answers to such questions from a local perspective (e.g., Cook et al., 2022), much is still unknown about such specifics from a global perspective. We believe this is a fruitful avenue for future research. Our efforts here provide (at least) one promising suggestion in this respect: Aspen's use of the global perspective and her associated remark that "it doesn't matter which point you pick" aligns with researchers' suggestions in the literature that robust reasoning about equivalence relations and classes involves a certain indifference towards the specificity of elements in particular classes (e.g., Asghari, 2019). This could be a useful refinement of our characterization of a global perspective. Another limitation of this study is that we have only investigated the reasoning of advanced mathematics students who had already developed the capacities needed to reason productively with equivalence relations. How students might, for example, develop the capacity to employ and move flexibly between the local and global perspectives of equivalence relations remains unclear. Working with undergraduate students in similar problem contexts might provide insights into how productive reasoning about equivalence relations *develops* rather than only describing what it might entail.

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