Coping With Coercion in Logic: A Case study

Alan Pasos
Simon Fraser University

Rina Zazkis
Simon Fraser University

The ability to make a logical inference is at the heart of mathematical experience. However, a personal reasoning does not always follow the rules of formal logic. In this case study we focus on responses of one participant, a secondary school mathematics teacher, to scenarios involving coercive logic (the construct is defined and exemplified in the paper). This allows us to take a detailed look into what guides and what influences the participant's approaches to various scenarios. We argue for the relevance of coercive logic to mathematics education, in particular, in exploring mathematical reasoning of students.

Keywords: logic, reasoning, coercive logic, mental models, illusory inference

Logical reasoning is inseparable from mathematics. However, research has shown that the study of formal logic does not always result in improved logical derivations (O'Brien, 1973; Wason & Johnson-Laird, 1972; Johnson-Laird & Byrne, 1991; Inglis & Attridge, 2016). Moreover, while there are disagreements within psychological reasoning theories, many theoretical perspectives that explain human reasoning are strongly linked to mathematical theories, such as probability, formal logic, or model theory, to mention a few.

In what follows we situate our investigation within ample research on mathematical reasoning and logical inferences. We then introduce the notion of coercive logic and analyze responses of one participant to scenarios involving coercive logic.

On Logic and Reasoning in Mathematics Education

Numerous studies have explored diverse aspects of formal logic in mathematics education. There are studies that concentrate on negation (Antonini, 2001; Barnard, 1995), logic connectives (Dawkins, 2017; Dawkins & Cook, 2017), axioms (Dawkins, 2018), contrapositive reasoning (Stylianides, et al., 2004), and quantifiers (Dawkins & Roh, 2020), among others. A general aspect of this research is the advocacy for the understanding of formal logic as a tool that supports mathematics learning.

One of the fundamental aspects of logic and reasoning is implication (Dawkins & Norton, 2022; Dawkins, et al., 2021; Hub & Dawkins, 2018), that is, conditional statements. As mathematical theorems, or propositions in general, can be presented in a form of conditional statements, students' understanding of implication attracted interests of researchers interested in learning proof.

Conditional reasoning has an extensive literature in the cognitive sciences (Nickerson, 2015). In particular, research surrounding the Wason's selection task (Wason, 1968) showed that not all reasoning behind deduction or inference belongs to the formal logic realm; there is a lot of deduction that is context dependent (e.g., Cheng & Holyoak, 1985). Further, mathematical background of participants influences their decisions, as noted by Stylianides et al. (2004) in their study of on interpretation of contrapositive statements, in comparing everyday scenarios and the related symbolic form.

Introducing Coercive Logic

The concept of coercive logic was developed and popularized by Raymond Smullyan (1997) in his book “*The Riddle of Scheherazade and Other Amazing Puzzles, Ancient & Modern*”. Smullyan (1997) defines coercive logic as “...an almost magical quality in that it forces [someone] to do something [they] wouldn’t otherwise have done...” (p. 256).” (He credits the name to his son-in-law Dr. Jack Kotik.)

Example of Coercive Logic

Smullyan (1997) introduces coercive logic with a story of Scheherazade (a character from the Arabic collection of stories known as “One thousand and one nights”), who every night escapes the execution by entertaining the Sultan with stories. Smullyan changes the stories to puzzles, where Scheherazade gets the Sultan’s promise to answer truthfully to one question, indicating only YES or NO. What would be the question that can spare her life? One of the suggested solutions for this logical puzzle is for Scheherazade to ask: “Will you either answer NO to this question or spare my life?”. Note that the NO answer implies that neither of the two alternatives holds, but this would be the false answer, which negates the Sultan’s promise to answer truthfully. As such, the only truthful answer is YES, which means that at least one of the two alternatives holds, and it must be the latter. With such a question Scheherazade can coerce the Sultan to spare her life.

Analyzing the case with the formalism of mathematical logic, we use:

N = you answer is NO ; L = spare Life

So, the given condition is “N or L”, which is true if at least one of the alternatives is true.

Let us examine the outcome of each possible answer. The NO answer negates both N or L, so we get “$\neg$ (N or L)”, which is equivalent to “$\neg$ N and $\neg$ L”, which is a contradiction.

As such, the YES answer is the only option to avoid contradiction, because in “N or L”, N is false, thus L must be true for “N or L” to obtain a valid true value.

To the best of our knowledge, there is no research in mathematics education that explicitly attends to coercive logic.

Coercive Logic, Brainteasers and Unfamiliar Environments

Paradoxes, puzzles, riddles, brain teasers, conundrums, etc. (basically any instrument that could create uncertainty or cognitive conflict) can enrich teaching and learning of mathematics (Kleiner & Movshovitz-Hadar, 1994). They have been part of mathematics and mathematics education throughout history. Scenarios that involved coercive logic have features that are similar to other kinds of recreational mathematics.

Coercive logic has characteristics related to cognitive conflict, inference, and uncertainty. These moments of doubt are opportunities to learn, teach, and learn as a teacher. Zaslavsky (2005) argued in favour of awakening certain dynamics that occur when dealing with cognitive conflict. She further mentioned that uncertainty is relevant to develop explanations and in general contribute to learning mathematics. We observe that coercive logic has the potential to become a motivational tool for the study of mathematical logic.

Scenarios of coercive logic, often presented as puzzles, are related to logic paradoxes, but there is an important difference. A paradox presents a self-contradictory statement or a situation that leads to contradictory conclusions. For example, in the famous Russell’s paradox (The barber is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself?), a barber who does not shave himself, actually shaves himself. However, in the coercive logic scenarios there is a “way out”, that is, a possibility of introducing an appropriate statement that will avoid a presumed paradox and lead to a desired outcome.

Research in mathematics education has demonstrated that asking learners to operate in unfamiliar environments can reveal their understanding of the elements of more familiar situations. For example, asking students about fractional numbers written in different bases reveals their understanding of place value (Khoury & Zazkis, 1994, Zazkis 1999); asking students to work with non-Cartesian coordinates highlights their knowledge of the Cartesian coordinate system (Zaslavsky et al., 2002, Zazkis, 2008). In a similar way, we expect to learn about participants' reasoning in the unfamiliar environment of coercive logic. In particular, we are interested in aspects of participants' reasoning and logical inference that are revealed when they respond to scenarios involving coercive logic.

The Study

Participants in our study were secondary mathematics teachers enrolled in a master's program in mathematics education. Our data sources included two written response questionnaires, which included coercive logic scenarios, and a semi-structured interview. As an introductory case study, which is a part of larger research, we focus here on one participant, Sal, whose responses were detailed and featured his approaches and hesitations.

Coercive Logic Scenarios

For the purpose of our study, we define coercive logic scenario as a fictional story culminating in a set of two conditional statements with different consequents, and a viable sought-after-statement that nullifies one or both conditionals to reach a unique valid and desirable outcome. This statement introduces a coercion. Consider the following example, used in one of our tasks:

> *You were trapped by cannibals, and they want to eat you. Their chief, who is known as a perfect logician, tells you that you can speak your last words, which should indicate your choice* (this is the story).
> *If what you say is true, then they will cook you in water.*
> *If what you say is false, then they will cook you in oil.*
> (this is the set of two conditionals)

The sought-after-statement is: *You will cook me in oil.* This statement nullifies both consequents: you cannot be cooked in water because the statements is not true. You cannot be cooked in oil, because "cook you in oil" is an implication of the false statement, and your statement is not false. In our design of coercive logic scenarios, we have constructed a variety of stories with sets of conditionals, in which the sought-after-statement leads to coercion and desirable outcomes. The participants were asked either to evaluate the given statements, or to formulate a statement that would lead to a desired outcome, such as avoiding a penalty or obtaining a reward.

Data Collection

The data collection was based on two questionnaires, administrated two weeks apart, and a semi structured interview scheduled after the completion of questionnaires.

Questionnaire 1 aimed at establishing a basic familiarity with formal rules of inference, and then introduced coercive logic scenarios. It included 6 multiple choice items. All items in Questionnaire 1 asked participants to explain their choice briefly for each case.

The first 4 items focused on conditionals with modus ponens structure, as well as ‘and’ and ‘or’ operators. In particular, acknowledging the differences in interpreting these operators in natural language vs. mathematical logic (exclusive vs. inclusive ‘or’; consecutive vs. symmetrical ‘and’), we were interested in participants’ choices as the basis for their familiarity with formal rules of inference.

For example, Item #3 presented the following statements: (1) All flowers are red or beautiful, and (2) All Tulips are flowers, followed by questions: (i) Are all tulips red? (ii) Are all tulips beautiful? (iii) Are all tulips red or beautiful? (iv) Are all tulips red and beautiful? Item #4 presented the following: Your dinner includes an alcoholic drink or a mock cocktail with your meal. Are you having only one drink with your meal? For each question the participants had to indicate (a) Yes, (b) No, or (c) Not enough information.

Items #5 and #6 introduced coercive logic scenarios. They presented a story followed up by two conditional statements with different consequents, and a sought-after-statement to achieve the desired outcome. The participants were asked to examine the outcome of the statement. In particular, Item #5 included the Cannibals scenario presented above, with the sought-after statement “You will cook me in oil”. The participants were asked to indicate (a) Yes, (b) No, or (c) Not enough information, to the two questions – Are they cooking you on oil? Are they cooking you in water” – and explain their reasoning. Item #6 included a scenario isomorphic to the Cannibals, but in a different (non-violent) context.

Questionnaire 2 included four coercive logic scenarios. Unlike items #5 and #6 in Questionnaire 1, in which the sought-after-statement was presented for participants’ evaluation, the first two scenarios presented multiple choice options, only one of which included the sought-after-statement that results in coercion. In the next two scenarios the participants were asked to formulate by themselves the sought-after-statement that would reach the desired outcome. As an example, Fig 1 presents the second item of Questionnaire 2.

Scenario 2: Hogwarts

You are walking through the halls of Hogwarts School of Witchcraft and Wizardry when you suddenly bump into Professor Severus Snape making him drop all of his books and potions. He then tells you that as a punishment, he will put a spell on you. He will give you the chance to make a statement to choose what spell he’ll curse you with.

You will make a statement such that

- if what you say is true, then the spell will be Petrificus Totalus (A body-binding curse that causes a person to experience temporary full-body paralysis.), and
- if what you say is false then the spell will be Obliviate (a spell that will make you forget all the content of your Witchcraft courses).

Then you have to choose one of the following statements to tell Professor Snape:

- Statement 1: You will give me the Petrificus Totalus.
- Statement 2: You will give me the Obliviate.
- Statement 3: You will either put a spell of Petrificus Totalus or of Obliviate.

Please chose one of the statements and explain the outcome of your choice.

Figure 1: Example of coercive logic scenario from Questionnaire 2

In an individual **semi-structured interview** setting selected participants were asked to examine and elaborate further on their written responses. The interviews lasted about 50 min; they were audiotaped and later transcribed. Participants were provided with an opportunity to change their responses and to reflect on their experience of addressing the questionnaire.

Theoretical Framing: Mental Models

The construct of "mental model" was initially introduced by Craik (1943) as mental representations of situations, which can be real, hypothetical or imaginary. Such models may contain perceptual qualities and may be abstract in nature. Johnson-Laird (1983) extended this idea to create a *mental model theory* of reasoning and deduction (Johnson-Laird & Byrne, 2002) Johnson-Laird (2012) summarised the three principles of the theory developed in the previous studies: (1) The principle of *iconicity*: A mental model has a structure that corresponds to the known structure of what it represents, (2) The principle of *possibilities*: Each mental model represents a distinct possibility, by capturing common features in different ways of occurrence, and (3) The principle of *truth*: Mental models focus on what is possible or true in the premisses. That is, what is impossible or deemed false is not commonly represented.

Mental model theory led to debates in the reasoning and cognitive sciences, as it advocates strongly for an effective prototype of inference non-dependent of formal rules. This prototype was able to predict systematic fallacies that cannot be explained by theories based solely on valid rules of inference (such as mental logic). In particular, Johnson-Laird & Savary (1999) introduced the notion of *illusory inferences*, that is, inferences that are compelling but invalid.

Example: Suppose that the following claims apply to a specific hand of cards, where only one of the two is true.

(a) If there is a king in the hand then there is an ace in the hand, or else,

(b) If there is a queen in the hand then there is an ace in the hand.

There is a king in the hand. What, if anything, follows?

Note that the conclusion: "There is an ace in the hand", is an invalid inference that is predicted by the mental model theory, based on the principles of possibility and truth. Given an exclusive disjunction of the two conditionals, (a) or (b), but not both, must be the case, however the further assertion – that there is a king in the hand – is definite. However, the disjunction is compatible with the falsity of one of the two conditionals. Since which one is false is unknown, there is no guarantee that there is an ace, even though there is a king. In particular, if (a) is false, the possibility of "ace in the hand" cannot be confirmed.

These illusions arise from the principle of truth. That is, the mental model for individuals who erroneously concluded "There is an ace in the hand", did not attend to the possibility that (a) in the premises can be false. It is shown that inferences that attend to truth only alleviate the illusions (Goldvarg & Johnson-Laird, 2000, 2001; Tabossi, et al., 1999).

Furthermore, according to mental model theory, people have a tendency to reason their way to consistency, that is, if they detect an inconsistency, they try to determine which premise to abandon to create consistency within a situation (for model accounts of these processes see e.g., Johnson-Laird, et al., 2000).

Attending to the case of Sal and his logical inference in considering coercive logic scenarios, we focus our research question on the particular case study: *How can mental models account for Sal's logical inference when responding to scenarios involving coercive logic*?

Results and Analysis

As mentioned above, our analysis is a case study of Sal. We chose Sal as his arguments appeared detailed but inconsistent, and, as described below, he provided unexpected ideas related to the coercive logic scenarios.

Item #4 on Questionnaire 1 is a specific question to evaluate the distinction between inclusive and exclusive “or”. Sal’s answer (Yes) to this question (having only one drink) points to Sal’s understanding of disjunction as an exclusive ‘or’ in natural language. However, Sal’s response to Item #3-iv (are tulips red and beautiful) was “Not enough information”, which is inconsistent with exclusive disjunction. Further, for Items #5 and #6 of Questionnaire 1, which introduce scenarios of coercive logic, Sal indicated “not enough information” for each of the two questions in both items. Sal was not able to resolve the scenarios, and this can be further appreciated in his comment “we can’t test whether the statement is true or not?”. It appears that Sal was seeking an a-priori indication of the truth value of the claim in order to be able to examine the implication, rather than proceeding with hypothetical assignment of truth. Without such knowledge he was not able to proceed to examining the conditionals.

In Questionnaire 2 Sal chose the option that included the ‘or’ clause when it was one of the available choices. That is, for Item #2 he chose the statement “You will either put a spell of Petrificus Totalus or of Obliviate.” Furthermore, he formulated the disjunctive statement when on the two items that invited participants to formulate a claim.

In the interview, Sal explicitly acknowledged being very confused on items #5 and #6 of Questionnaire 1. However, when asked to consider his responses to the items of Questionnaire 2, he noted, with respect to the Hogwarts scenario (see Fig. 1):

Sal: But by saying this entire thing, it becomes true though, like it cannot be false [points out statement 3], this cannot be false is what I would like to say, cause whatever I say I am either gonna get either PT [refering to Petrificus Totalus] or Obliviate, right? But then the entire statement is true. So, I thought I would be getting the PT spell.

Interviewer: Ok, how was that you come to the idea that this one is true then the second one [pointing to the second conditional] is not happening, right?

Sal: Right, yes, that's right the second one, so I won't be getting the Obliviate if I say statement 3, it's what I was thinking.

The selection of disjunctive statement, which is tautological but true, invalidates one of the conditionals in the presented scenario. That is, the second conditional becomes irrelevant, yet the exclusive disjunction between the conditionals holds valid.

A disjunction in which components hold opposite truth values produces a tautology. Nevertheless, a choice of tautology resolved the situation for Sal, as he considers the consequent associated with truth as a preferred choice. That is, PT is preferable as it is temporary.

Sal: Oh yeah, so, this PT spell says temporary full body paralysis something that's temporary. My paralysis will be gone, that's what I was thinking, versus this obliviate in that one you just forget all of the contents of the stuff that you learn from the course, right? So, I've rather had a temporary one rather than just forgetting everything that I've learnt. So, that's why I choose it, I'll go with the truth. Choose the PT one and say the truth.

This could have been an oversight in our design as we considered both choices to be undesirable and avoidable via appropriate logical coercion. However, for Sal the “truth” was associated with a desirable outcome, which is the less hurtful of the provided unfavorable choices.

During the interview Sal emphasized the fact that he wanted to choose the statement that contained a truth antecedent, and thus simplify the scenario to a unique conditional to deal with.

Interviewer: I can see that there's some logic over all in the general aspect of your answers, for example, you were congruent in the choices that you were making [...] in this scenario 4 [item #4 in questionnaire 2] that you were asked to give a statement, you actually selected something similar to the previous scenario. So, the question now is: how come? Why did you decide on a statement of a similar form to the previous ones?

Sal: If I had not read these first two questions, because it actually gave me the [pause] by saying "either or" I am making a true statement and can choose whatever I want to […] So, I wanted to get something off of the statement, so let's make a truth statement, so I think the first two question gave me a way of [pause], in my mind getting me definitive truth statements. So, then I can use that "either or" logic in here so that I am choosing the truth. So, I am making a truth statement […].

We note in this excerpt that the principle of truth manifested itself to increase the possibilities of a mental model that addresses the given logic scenario without the need to consider a possibility of a false statement. We consider Sal's responses as an example of *illusory inferences,* guided by the principle of truth.

But it is not enough to deal with one conditional to find a way out of coercive scenarios. This is clearer when there is no preference in the consequents of the conditional, such as in Cannibals scenario. It is the independent evaluation of both conditionals that allows participants to formulate or choose an appropriate sought-out-statement. Sal was not able to reason on items in which he did not identify a desirable outcome. However, when a choice was possible, he corroborated a mental model principle of truth that did not result in coercion, but allowed for a desirable (that is, less hurtful) choice.

Considering Sal's written responses, and his elaborations during the interview, we conclude that the principle of truth guided the construction of the mental models that Sal used to resolve the coercive logic scenarios. This guidance was effective in the scenarios in which there was a preference for an outcome within the coercive logic scenarios, but failed when it was necessary to construct dual mental models to deal with two conditionals presented in the scenarios.

Discussion and Future Research

Our research aimed at studying participants' reasoning and logical inference when they respond to scenarios involving coercive logic. In particular, we focused on the case of Sal and identified the prevalence of the truth principle in his mental models. Our contribution can be seen in introducing coercive logic scenarios as a method for studying participants' reasoning that may not be revealed by other means.

A limitation of the study is in our initial design of tasks that requires a minor adjustment. Our design of scenarios progressed from (a) asking participants to evaluate a sought-after-statement in Questionnaire 1, to (b) choose an appropriate claim from the presented possibilities, and (c) to formulate a claim in Questionnaire 2. However, the presented tautology and the disjunctive claim in (b) has influenced Sal's reasoning and directed his subsequent choice to an option that is "always true" and in such avoided coercion. We will eliminate such an option as we continue our research.

Our future research will explore additional theoretical constructs that may explain Sal's (and other participants') reasoning, either intertwining with mental models theory or presenting an alternative. For example, how can the notion of reducing abstraction (Hazzan, 1999; Hazzan & Zazkis, 2005) explain Sal's approach, especially the prevalence of the truth principle?

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