Undergraduate Students' Models of Subtraction and its Significance for Calculus

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The operation of subtraction, though emphasized at the elementary level, has a central role in expressing key ideas in Calculus. In graphical representations of Calculus concepts, a magnitude interpretation of a difference, that b–a gives the distance between a and b, often supports students in making sense of the representation. A magnitude interpretation uses a determine the difference model of subtraction, in which subtraction find how much a differs from b. Using exploratory teaching interviews, we investigated the models of subtraction that nine undergraduate students used on tasks designed to support them in using a magnitude interpretation. We found four distinct models of subtraction that students employed in these tasks. We describe these models and associated observable behavior. We discuss implications of these findings for the teaching and learning of Calculus and future research directions.

Keywords: subtraction, graphical representations, Calculus, difference expressions, number lines

One way to conceive of the subject of Calculus is as the study of change. As such, the operation of subtraction, which can be used to express the change in values of a variable, figures prominently in Calculus. Difference expressions form the backbone of expressions in the limit definition of the derivative ($f(x+h)–f(x)$), the Mean Value Theorem ($f(b)–f(a)$), and the Fundamental Theorem of Calculus ($F(b)–F(a)$), among others. The operation of subtraction may be conceptualized in numerous ways, including: the takeaway model, the determine the difference (comparison) model, the part-whole model, and adding up. In particular, the determine the difference model, in which subtraction is used to find how much one amount differs from another, plays a central role in Calculus, to find the change in two values of a given variable. Researchers have highlighted the role of the determine the difference model in Calculus, and some have also argued that introducing Calculus concepts, such as displacement and net change, may motivate and necessitate this model of subtraction, especially involving negative values (Stroup, 2005). When considering graphical representations in Calculus, changes in quantities or values of a variable are often represented by horizontal and vertical segments. When students can associate a difference with the distance between two positions, they can express the length of the segment as a difference and connect such expressions with segments in a graph, what Parr (2023) refers to as a magnitude interpretation of a difference.

As part of a larger study, we conducted exploratory teaching interviews (Steffe & Thompson, 2000) with nine undergraduate students using an activity designed to support students in expressing distances between two functions in the Cartesian plane using differences (Parr & Lippe, 2024). Expressing such distances are necessary for setting up integrals to evaluate areas and volumes in Integral Calculus. In this study, we ask: *What models of subtraction did undergraduate students use on an activity designed to promote a magnitude interpretation of difference expressions?*

Empirical & Theoretical Background

We begin by describing two models of subtraction commonly introduced at the elementary level, the takeaway model and the determine the difference model (van den Heuvel-Panhuizen &Treffers, 2009; Selter et al., 2012), also called the comparison model (Usiskin, 2008). The takeaway model of subtraction involves conceiving of subtraction as beginning with one amount of a quantity, *b*, taking away some amount of that quantity, *a*, which results in a final amount of the quantity. In this model, the result of subtraction finds this remaining amount. The determine the difference model involves conceiving of two amounts, *a* and *b* simultaneously, and then comparing how much larger *b* is than *a*, to find how much *b* differs from *a*. Research has shown that students struggle with such comparison operations compared to other types of subtraction problems (e.g., Stern, 1993). Further, students often default to using a takeaway model for subtraction, even when a determine the difference model provided more computational efficiency. (e.g., Baroody, 1999; Stern 1993).

To consider how a determine the difference model for subtraction may be used to represent distances graphically, we use Parr and Lippe's (2024) description of the components of a magnitude interpretation. In this description, a magnitude interpretation (Parr, 2023), involves combining the determine the difference model with a conception of distance as a quantity in space. The magnitude interpretation combines these such that a difference expresses the distance between two positions in one-dimension. To illustrate why the difference x_2–x_1 gives the distance between these two positions, Figure 1 shows x_1 and x_2 themselves represented as magnitudes, or distances from zero. The expression x_2–x_1 may then be considered to be composed of magnitudes (Parr et al., 2021; Parr & Lippe, 2024).

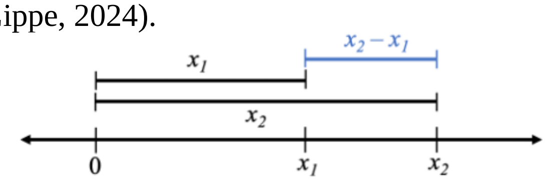


Figure 1. Composed magnitude interpretation of the expression $x_2 - x_1$ (Parr & Lippe, 2024, p. 507)

In the activity that students worked through in this study, students were encouraged to use a magnitude interpretation of differences in several ways, including drawing segments from zero to represent a given value, as in Figure 1. The magnitude interpretation of a difference expression, which aligns with a determine the difference model of subtraction stands in contrast to a cardinal interpretation (Parr, 2023) of a difference, which aligns with a takeaway model. When students use a cardinal interpretation, they often count units, such as tick marks, to show what a difference represents graphically. Even at the undergraduate level, students may favor a cardinal interpretation (Parr, 2023), perhaps because a takeaway model of subtraction is favored in their experience in elementary mathematics and daily life (Selter et al., 2012). In a previous study, we found that less than 40% of undergraduate Calculus students could correctly express a horizontal segment between a function and the vertical line x=2 using a difference expression in terms of x (Parr & Lippe, 2024). These findings motivated the development of the activity with a goal to better understand how to support students in using differences to express distances in graphs of functions in the Cartesian plane for their work in Calculus.

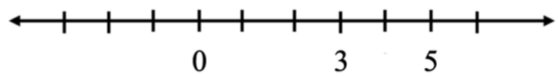
Methods

We conducted exploratory teaching interviews (Steffe & Thompson, 2000) with nine undergraduate math or science students at a small private college in 2023. In the interviews, students were asked to work through tasks in the Interpreting Graphs for Calculus Activity (Parr

& Lippe, 2024). The first task asked students to measure with a ruler, various lengths on a number line a certain distance to the left or right of zero, to support students in conceptualizing distances as measurable quantities within number lines. The second task, shown in Figure 2, asked students to draw a segment with a length of 5 on the *x*-axis starting at zero, a segment with a length of 3, and then a segment to represent the difference.

2. Draw the segments described using the *x*-axes provided.

a. Draw a segment with a length of 5 above the *x*-axis below, beginning at 0.
b. Draw a segment with a length of 3 above the *x*-axis below, beginning at 0.
c. Draw a segment to represent the difference between the two segments you've drawn on the *x*-axis below.



d. What is the length of the segment you drew in part c.?
e. Write an expression (not a single number) to represent the difference between the two segments.

Figure 2. Task 2a-e in the Interpreting Graphs for Calculus Activity promoting a magnitude interpretation

The subsequent tasks 3-7 asked students to represent difference expressions using segments on horizontal and vertical number lines, or to write a difference expression for a segment shown. These difference expressions began with two positive integers, then included negative integers, non-integer values, and finally variables. A member of the research team served as the interviewer, with the first author serving as a witness if not the interviewer themselves. In each interview, the interviewer asked the student to further explain their thinking and may have posed additional questions to probe and test the boundaries of the students' current understanding of the task (Steffe & Thompson, 2000). The tasks were designed to promote a magnitude interpretation of a difference expression, but the interviewers refrained from expressly telling students to use this type of interpretation, unless it seemed like the student was unable to overcome a perceived obstacle without such intervention. We video recorded the interviews, as well as the iPad screen of students' work on the activity to capture their written work in real-time. The data for the study included students' words, gestures, diagrams, and written work on the activity.

We analyzed the data, with students' reasoning in the moment as our unit of analysis. That is, we considered students' models of subtraction as subject to change within their work on a given task and did not limit our description of students' work on a task to a single model. At the beginning of the data analysis process, we were attuned to the two distinct models of subtraction described above: a takeaway model and a determine the difference model. The research team used a grounded approach (Corbin & Strauss, 2014) to analyze the interviews, beginning with describing their model of how a student was reasoning in each task of the interview. At this stage, some details in students' activity became apparent, including activity using segments in distinct ways, such as moving segments and comparing segments visually. Through open and axial coding (Corbin & Strauss, 2014) we identified four distinct models of subtraction.

Results

In this study, we report four distinct ways in which the students we interviewed modeled subtraction on tasks designed to promote a magnitude interpretation of difference expressions on horizontal or vertical number lines (axes). We refer to these four models as: (a) magnitude, (b) takeaway values, (c) a takeaway segments, and (d) translation. We describe each model and the observable behaviors we used to categorize the students' activity as the given model.

Magnitude Model

We observed several students model differences as composed magnitudes, which aligned with the interpretation we sought to promote in the activity and teaching interview. Due to space, we only describe the model and observable behavior in Table 1 at the end of the section.

Takeaway Values Model

Several students modeled differences as taking away values on the number line. In these instances, students began at a position *b* on the number line, went back (or down) *a* units, and arrived at position *b*–*a*. This takeaway values model aligns with conceiving of subtraction as takeaway, more broadly, and with a cardinal interpretation of differences. We found that students used this model even after using what appeared to be a magnitude model on prior tasks.

We provide the example of the case of Ellie. Ellie had used a magnitude interpretation on previous tasks in the activity. However, on Task 5a, Ellie used what we infer as a takeaway values model. Ellie was asked to draw a segment representing *x*–2 on a horizontal number line. On this number line, *x* was represented as a point near 4.5 and labeled "*x*" (see Figure 3). These episodes show her figuring out how she would represent the result as a line. She knows the result of subtraction is a single number, so she decides that the final line she draws will be from *x* to the final product of subtraction, which in this case would be about 2.5. Here, she is using her line to represent the numerical value 2 being taken away from the starting point *x*.

Ellie: (*pauses, draws a dot near 2.5 in blue*) And by segment you really want the line?
Int: Yes.
Ellie: Okay. (*Pauses, draws line from dot at 2.5 to dot at x*)
Int: Can I ask why you were hesitant?
Ellie: Because to me *x*–2 is just a point. Like I would just take this *x*, which is…like, three, four, four and a half, and then subtract two and I would go down to two and a half. So like to me I wouldn't draw like a line segment to represent that value….I want my leftmost endpoint to represent what *x*–2 would actually be in terms of numbers…
Int: Okay so your segment is…representing, what, the answer to *x* minus 2?
Ellie: Kind of? Because I'm like literally going 2 down from *x* is what I feel like my segment is representing. Because I don't know really how else to express a single point, or at least what I'm interpreting as a single point, as a segment.

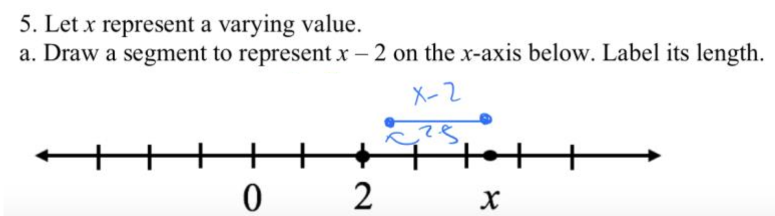


Figure 3. Ellie's work on task 5a, in which she modeled x–2 as the result of beginning at x and moving left 2 units.

In this episode, Ellie was reluctant to represent the difference as a segment, when in her mind it was "a single point." She does decide to draw a segment when she clarified the directions of the task and explained that she was going "down to two and a half." Notably, she used 4.5 as an approximation for the value of *x*, which was intended to be a generic varying value as stated in the task. Ellie's final line was a representation of the numerical value removed from *x*. By using 2 as the length of her final segment instead of one of the segment's endpoints, she suggests that she sees it as a value taken away from *x*. Therefore, we infer that she is using a takeaway values model here. For the students using a takeaway values model who were asked to draw segments, the segments were a record of the movement from one value to another, rather than a distance whose value was the resulting difference.

Takeaway Segments Model

Distinct from the takeaway values model described above, in which students represented taking away values as counting back some amount, we found some students use a takeaway model of subtraction, but represent this through operating on the segments themselves. That is, they modeled subtraction as taking away part of a segment from another segment, leaving a segment as a result. We provide an instance with a student, Chloe. The interviewer asks Chole to explain her thinking for task 6a. She begins to explain this using the number line provided in the task, but then, she switches to thinking about how this could relate to an example about eating a chocolate bar.

Int: How would you explain for [part] a why the segment represents 3.5 minus 2?

Chloe: If you have a bigger number and you take away a smaller number, the smaller number is going to be canceled out completely, but a little bit of the big number will be left. You view almost… 3.5 as a whole, and you're taking a part out of it, and the part would be 2. So, then the remainder left would be just this tiny segment between 3.5 and 2.

Int: Is there a way you could represent what you were envisioning with a whole and a part using segments on the graph?

Chloe: Yeah. So, if I have 3.5 (*draws a rectangle next to the number line from 0 to 3.5*), and this would be 2 right here (*draws a line inside the rectangle in line with the 2 on the number line*). Okay. I'm gonna pretend that it's an object, like a candy bar. And let's say, if I only eat 2 segments out of the 3.5 candy bar, all of this will be erased right here (*scratches out the part of the rectangle under the line at 2*), and then I'm left with a segment, this red segment (*colors the remaining part of the rectangle red*)…since there's a larger number subtracted by a smaller number, I'm kind of, I'm viewing it as the larger number will stay more intact, the little number will get eaten up completely.

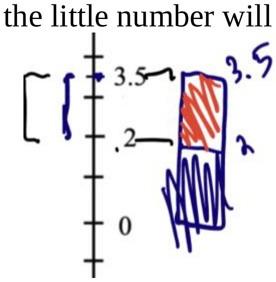


Figure 4. Chloe's work on task 6a

In this episode, we infer that Chloe was using a takeaway segments model of subtraction. Chole viewed the segment between 3.5 and 2 as what is remaining when you take away the segment from 0 to 2 from the segment from 0 to 3.5. She referred to 3.5 as "the whole" and the 2 as the "part" that is taken out. Chloe further emphasized that she was operating on these segments when she introduced the idea of a physical object, a candy bar. Although Chloe still appeared to envision a process of taking away, we note that the activity associated with this model is distinct from the takeaway values model, in which a student may begin at 3.5 and count back 2 units. We note that this model broke down for Chloe when she tries to apply it in task 6b, which asks her to represent 6 – (–2). She could not figure out how to apply the takeaway segments model and remove a negative segment from a positive segment because there was no overlap between the two segments.

Translation Model

Finally, we observed students using what we refer to as a translation model of subtraction. In these instances, students found a numerical result, and then represented that result as a length from zero. We give the example of Yasmine on task 4c. Yasmine drew a segment labeled "a" for –1, a segment labeled "b" for –5 (we note that she drew this to the right of zero rather than to the left) and then drew a segment from 0 to 4 which she labeled c, in response to task c, which asked her to draw a segment to represent –1–(–5). In this way, Yasmine translated the distance between –5 and –1 and shifted this to the right, to another segment with a length of four. Although Yasmine did not attend to the distance between –5 and –1, we refer to this as a translation model, as the drawn segment has the length of the difference, yet the endpoints of the segment have shifted. When asked to explain her reasoning, she shared that she knew the numerical result of the difference was four and that she was "more focused on getting the length right" rather than thinking about the relationships between segments a, b, and c on the *x*-axis.

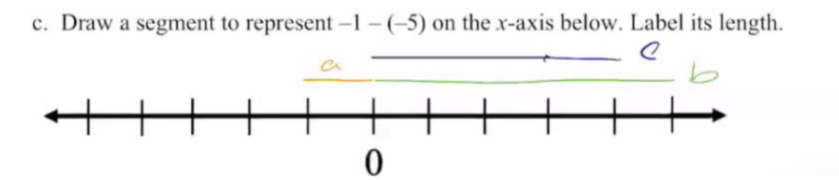


Figure 5. Yasmine's work on task 4c, an example of a translation model of subtraction

In Table 1, we summarize our description of the four models we identified and the observable evidence we used to help classify student's reasoning as such.

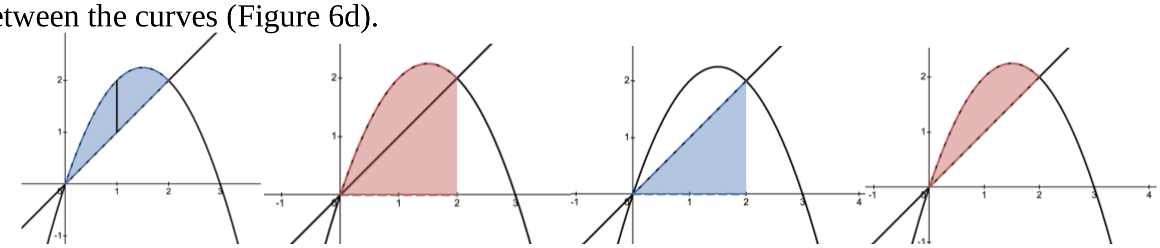
Table 1. Four Models of Subtraction

	Models *b–a* as	Observable Evidence
Magnitude	The distance between positions *a* and *b*	• Language: "length," "distance," "from *a* to *b*" • Draws a segment from *a* to *b* and labels it *b–a* • Attends to the endpoints when referencing segment
Takeaway Values	The resulting position starting at position *b* and moving left (down) *a* units	• Language: "takeaway *a*," "go *a* back" or "down from *b*" • Starts at position *b*, counts *a* units back, labels final position as *b–a* • Segments, if drawn, record movement from one position to another (length of segment may be associated with *a*). • Counts discrete units, uses counting language
Takeaway Segments	The resulting segment from removing a segment of length *a* from a segment of length *b*	• Language: "the segment *b–a*," "remove," "piece" •Considers segment from 0 to *b*, removes segment from 0 to *a*, remaining segment is result • Emphasizes remaining segment as result, may not attend to length of segment, or endpoints of the segment
Translation	The distance between *b–a* and 0	• Language: "length" • Finds result, draws segment of that length from 0 to *b–a* • Does not require end points of the segment to be *a* and *b*.

Discussion

In our study, we identified four distinct ways in which undergraduate students modeled a difference expression of the form $b–a$ in a single-dimension. These four distinct models emerged during tasks designed to promote a magnitude interpretation of $b–a$ as the distance between positions a and b. We noticed trends in the ways in which students used their models to explain their reasoning, and the extent to which students could see past their models. We theorize there may be a connection between the distinctions in figurative and operative imagery (Moore, 2016; Thompson et al., 2024) and the ways in which students were constrained or empowered by their models for modeling a range of differences, including those with negative integers, non-integer values, and variables. For instance, the takeaway segments model appeared to be tied closely to the models themselves, with students operating on an object within the model. For Chloe, these models were not able to be adapted to account for a difference between a positive and negative value. We plan further research in this direction, as well as understanding when and how students adapted their models throughout the activity.

Although three other models of subtraction emerged in a context where we were attempting to promote a magnitude model, we see ways in which some of these models can be leveraged in the teaching and learning of Calculus. We conclude by commenting on the role of various models of difference expressions in one example from Integral Calculus, area between curves. The area between graphs of functions f and g in the Cartesian plane can be found using integrals of the difference of two functions. The justification for why the difference of the functions finds the area between the curves can be conceived in several distinct ways. For example, consider the functions $f(x) = x$ and $g(x) = –x(x–3)$. Locally, the difference of the two outputs of the functions for a value of x, $f(x) – g(x)$, gives the height of a generic rectangle in the area between f and g (Figure 6a). Another way to conceive the area between the curves is as the global difference between the area under f (Figure 6b) and the area under g (Figure 6c) resulting in the area between the curves (Figure 6d).



(a) (b) (c) (d)

Figure 6. Pointwise difference gives height of any rectangle between f and g (a) vs. Global difference of area between functions gives difference between area under individual functions (b), (c), (d).

We note similarities between this global difference of the area under two functions and the takeaway segments model we observed in our study. The global difference of the area under two functions, and the area between two curves considered pointwise may each be used to conceive of the area between functions. If a student can readily perceive of these illustrations as related to the integral expression, these visuals together can serve as a proof of the difference property of integrals, that the integral of the difference of two functions is equal to the difference of the integral of each function. Still, a third way to conceive of why an integral of a difference of two functions gives the area between the curves is to conceive of a difference of two functions as translating the area to be between one singular function and the horizontal axis. In our example, the equivalent area would be under the function $h(x)=(3x–x^2)–x$. We anticipate connections between the translation model we observed in this study and this third way of conceiving of how integrals of differences of functions find areas between curves.

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