Students' Actions Related to Complex Exponents

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Interest has been growing in how students learn concepts in complex analysis (Troup, 2015; Hancock, 2019). A concept generalized in complex analysis is exponents and logarithms, yet little is written on how students leverage their existing knowledge to the complex case. In this paper, I present data from task-based interviews via the lens of APOS theory to present two students who leveraged their knowledge of real-valued exponential and logarithmic operations to varying degrees. I present data from a single question asking them to compare the magnitude of complex numbers and argue that their schemas for the two subjects were separate.

Keywords: Complex Analysis, Exponents, APOS Theory

Many studies concerning how students understand exponential or logarithmic operations and functions have looked at how students generalize the domains of exponents from natural numbers to rational numbers (Elstak, 2007; Williams, 2011). In the context of complex analysis, there is a notion of raising a complex number to a complex power, defined as $z^w = e^{(w \cdot Log(z))}$, where $e^{a+bi} = e^a(cos\ (b) + i \cdot sin\ (b))$, and Log(z) denotes an inverse of e^z. At the time of writing, there is yet a study on how students understand such expressions, either from the literature on learning of exponents & logarithms or learning of complex analysis.

In this study, I extend studies on exponents and complex analysis that used APOS theory (Pitta-Pantazi et al. 2007, Tabaghi, 2007). Foundational among these studies is that from Danenhower (2000) which touches on the theme of Thinking Real, Doing Complex (T$\mathbb{R}$D$\mathbb{C}$). Extending his work, along with that from Troup (2015, 2019), Hancock (2018, 2019), and Soto (2014, 2016), I investigate the question "How do students leverage their knowledge of real-valued exponential operations to reason about the complex analog?"

Theoretical Framework and Literature Review

Next, I briefly review APOS Theory and the studies related to the learning of complex analysis, exponents, and logarithms situated within it.

APOS Theory

The four constructs of actions, processes, objects, and schemas constitute the basis of APOS theory. APOS theory, as described by Asiala et al. (1997) is concerned with the construction of schemas and mental objects, and the theory posits that objects are the result of sufficient reflection of processes or the thematization of schemas. According to Asiala et al. (1997), an action "is a transformation of objects which is perceived by the individual as being at least someone external," a process is the result of reflective abstraction on an action or the reversal of an existing process, and an object is a result of either "when an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that actions or processes can act on it, and can construct such transformations" or schemas via "thematization," which is defined as a schema's inclusion in other schemas.

Dubinsky (1991) and Asiala et al. (1997) define a schema: as a "coherent collection of objects and processes," (p. 166) and argue that the main concern of math education researchers should be the construction of schemas. Although a schema is internal to the individual, Dubinsky (1991) highlights the idea of genetic decomposition as "detailed descriptions of schemas in terms of these mental constructions" as "a way of organizing hypotheses about how learning mathematical concepts can take place" (p. 182). Although schemas, objects, and processes aren't possible to directly observe by outsiders, Dubinsky (1991) describes them as the entry point for reflective abstraction.

Exponents

Some of the earliest studies around exponents and logarithms were from Confrey (1994), Confrey & Smith (1995), and Weber (2002). The primary contribution of Confrey (1994) and Confrey & Smith (1995) is the theoretical notion of the splitting world and covariation as a means to productively understand the exponential function. Weber (2002) reports on the result of a teaching experiment primarily through the lens of APOS theory. He also uses Sfard's notion of reification and Sfard's postulation that structural understanding is "necessary to reason about mathematical concepts" (p. 1020) His test group was found to be able to "use their conceptual knowledge to reconstruct" exponent and logarithm rules. (p. 1025) Like Confrey (1994) and Confrey & Smith (1995), Weber finds that the idea of exponents as iterated multiplication is an insufficient model for generalizing to non-natural numbers.

Pitta-Pantazi et al. (2007) identify three levels of understanding exponents within APOS theory via written questionnaires and semi-structured interviews. The interviews consisted of asking students to compare the magnitude of exponential expressions. The first level they identify corresponds to "the pre-conceptual stage at which students are getting used to certain computational operations on the exponents with natural numbers," or an action understanding of exponents. The second level "corresponds to the operational stage at which students use the familiar processes on exponents with new kinds of numbers such as rational numbers," or is characterized by generalizing valid bases. In the case of rational numbers, it can also be seen as a coordination of applying exponents on the numerator and denominator. The third and final one "corresponds to the restructured stage at which students recognize exponents as a mathematical object" (p. 309). Here, "object" is used in the sense of APOS theory.

Complex Analysis

Danenhower's (2000, 2006) work is foundational to contemporary investigations into students' learning of complex analysis and is situated within a framework of radical constructivism and APOS theory. His primary hypothesis distinguishes the vector, cartesian, and polar forms of complex numbers and conjectures that students will have difficulty moving between these forms.

One part of his dissertation focused on the "multirepresentations of complex numbers" (p. 109), which is also how he thematizes his findings (p. 110). One of these themes was "Shifting Representations," which refers to how students change the representations of complex numbers to expedite (iterated) multiplication, division, or addition/subtraction. In such context, Danenhower describes four stages of understanding complex fractions in terms of APOS theory, the last one being "the ability to indicate when to shift representations" from "our conception of complex fractions" (p. 124). He also presents the following criteria for each stage of the theory.

Table 1. Danenhower's (2000, p. 125) findings on simplifying positive integer powers of complex fractions in APOS theory

Action	"Concerned with the details of the realizing algorithm for each problem. Applies algorithms routinely and apparently without thought of other

	possibilities (absence of evidence of a thought process leading to a decision to perform the calculations that were undertaken by the student).”
Process	“Shifts to the polar representation at some point, but is still primarily concerned with the details of the simplification. Possibly uses some "short cuts", e.g., uses (a + ib)(a - ib) = $a^2 + b^2$ without calculation. Generally fluent technique, i.e., is able to do the simplification correctly with few obstacles. Shows that they are doing the calculations by decision, rather than because ‘it is what you do in this kind of problem’”.
Object	“Uses prior knowledge freely. Indicates consideration of more than one method of solution. Uses geometry to aid calculations. Freely modifies realizing algorithm or polar method to save work.”

Similar findings regarding Shifting Representations would be replicated by Panaoura et al. (2006), Karakok et al. (2014), and Soto-Johnson & Troup (2014). Panaoura et al. (2006) found that "the identification of two distinct clusters of pupils revealed that there are two fundamentally different perspectives from which complex numbers are viewed and approached: the algebraic and the geometric approach" (p. 700). Soto-Johnson & Troup (2014) found that algebraic and geometric reasoning decreased as time went on with a bias toward algebra.

Data and Methods

This paper draws on a wider data set concerning how students understand the complex exponent and logarithm for a pilot study. In this paper, I mainly look at responses to questions concerning exponents based on items from Pitta-Pantazi et al. (2007) which looked at comparing the magnitude of exponential expressions, as well as responses to earlier questions based on those from Danenhower (2000) which give further context to their responses.

Research Design

Participants and Setting: Two participants, Monk and Thomas, were recruited from an American R1 university. These students were currently enrolled in an undergraduate complex analysis course and recruited with the aid of their instructor. The study was structured to have each participant participate in two individual interviews and two paired interviews, totaling six interviews, but this paper reports only data from the individual interviews.

Instrument: Complex exponents can be understood to require a coordination of three actions to achieve an object-level understanding: complex multiplication, the complex exponential function, and the complex logarithm, and so the item I present requires coordination of these actions. This paper reports on the result of a specific question based on Pitta-Pantazi (2007): Given expressions a, b, c, compare them in terms of “<,” “>” or “=.” The expressions were as follows:

1. $|(2e^{i\pi/4})^3|, |(2e^{i\pi/4})^4|, |(2e^{i\pi/4})^5|$
2. $|(-1+i)^{3+i}|, |(-1+i)^{4+i}|, |(-1+i)^{5+i}|$
3. $|(-1+i)^{3+i}|, |(-1+i)^{3+2i}|, |(-1+i)^{3+3i}|$

These items came after the participants calculated the following: $\frac{-2+2i}{(1+i)^3}$, $-\frac{8}{(\sqrt{3}+i)^6}$, $(3\sqrt{2}+3\sqrt{2}i)^{\frac{1}{2}}$, i^i, i^{2i}, $3i^{1+i}$, $(-(1+i)^{-5-3i})$, as it contextualizes their behavior for the three items and gave them opportunities to interiorize their relevant actions into a process for taking complex powers. Some of these items were chosen due to their use by Danenhower (2000).

Analysis

Although there were similarities between the two participants, there were also differences in how they reasoned about these three problems to such an extent that they reached different conclusions for question 3. Thomas deduced the sequence was decreasing while Monk deduced it was increasing. To understand how Monk and Thomas compared the magnitudes of exponential expressions, it is necessary to understand how they computed exponentials.

Calculating Real Exponents

Common between both participants was their strategy in computing the factors with purely real exponents. When faced with an exponential term in the denominator, they immediately multiplied by some power of the conjugate. To simplify $\frac{-2+2i}{(1+i)^3}$, they multiplied by $\frac{(1-i)^3}{(1-i)^3}$, and similarly, for the second they multiplied by $\frac{(\sqrt{3}-i)^6}{(\sqrt{3}-i)^6}$. And then when calculating the square root, $(3\sqrt{2}+3\sqrt{2}i)^{\frac{1}{2}}$, Monk deferred to the definition of $z^w = e^{w \cdot log(z)}$, writing $e^{\frac{1}{2}log(3\sqrt{2}+3\sqrt{2}i)}$, and then eventually simplified to $e^{\frac{1}{2}ln(6)}(cos\,(\frac{\pi}{4}) + i\,sin\,(\frac{\pi}{4}))$. In contrast, Thomas drew a picture of the vector $(3\sqrt{2}+3\sqrt{2}i)$ in the complex plane and reasoned that "the square root should stay on the same line, because the angle is retained... and the modulus is greater than 1 so the length should be shortened." Then he started to reason with the Pythagorean theorem, concluding "I wouldn't know how to expand it otherwise."

According to Danenhower (2000), they operated at an action-level understanding of complex numbers, characterized by their reliance on algebraic definitions and manipulation. Their responses to the square roots are stronger evidence of this claim, where according to Pitta-Pantazi (2007), they were operating on a Level 1 understanding of fractional exponents with a complex base. Monk relied solely on an algebraic calculation, and although Thomas attempted to use geometric reasoning to calculate the square root, he was ultimately unsuccessful and incorrectly stated that taking square roots leaves angles unchanged. So there was also a lack of evidence that complex multiplication was reversed for either participant. Thomas' assumption that square roots stay on the same line is possibly an instance of the T$\mathbb{R}$D$\mathbb{C}$ phenomenon. Thus, I conclude they have not achieved a robust process understanding of complex multiplication in general but were working with a process understanding of algebraic complex multiplication.

Calculating Complex Exponents

Monk: When tasked with calculating $3i^{1+i}$, Monk immediately deferred to the definition by writing e((1+i)(Log(3i)))=e(Log(3i))e(iLog(3i)).[1] When asked how he got the second term, Monk described a mental calculation of distributing the Log(3i) term and using the established property of $e^{z+w}=e^z e^w$. He continued to simplify by writing 3ie(iLog(3i)), using the inverse relationship between logarithm and exponent, and tried to "do [Log(3i)] in my head," subsequently simplifying to 3ie(i(ln(3)+iπ), and finally to 3e(-π)i(cos(ln(3))+i·s(ln(3))). In this inscription, "s(ln(3))" stands for "sin(ln(3))."

When asked for another means of calculation, he checked if the value he calculated prior would match with the identity $3i\cdot(3i)^i$ as an instance of the conjectured rule $a^{z+w}=a^z a^w$. He

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[1] At this point, Monk stopped writing exponential expressions with the power above the base, hence the change in style.

simplified $3i\cdot(3i)^i$ into $3i\cdot 3^i i^i$ using another conjectured rule that $(ab)^z=a^zb^z$, and used his past work to replace i^i with e(-π/2). Then, he replaced 3^i with e(iLog(3)) and had a final answer of 3e(-π/2)i(c(ln(3))+is(ln(3))). He recognized that these two expressions were not equal, due to a different factor of e(-π) and e(-π/2), but corrected it after the researcher pointed out that the argument of 3i is not π, but π/2. After this, he concluded that the rule $a^{z+w}=a^za^w$ was valid.

He immediately used this rule in calculating $(-(1+i)^{-5-3i})$, after first distributing the negative sign. By reducing the calculation to $(-1-i)^{-5}(-1-i)^{-3i}$, he looked to simplify the term $(-1-i)^{-3i}$, again by using the definition of complex exponents. He rewrote it as e(-3iLog(-1-i)) and then as e(-3i(ln$\sqrt{2}$-$\frac{3\pi}{4}i$)), and then by distribution of the -3i term, arrived at a final answer of $-(1+i)^{-5}$e(-$\frac{9\pi}{4}$)e(-3i(ln$\sqrt{2}$)).

His performance demonstrated a strong action understanding of complex multiplication, but as evidenced by his calculation of the square root, he has not reversed iterated multiplication. His initial claim that the argument of $3i$ as π reveals his attempt to conceive of the logarithm as a process, and his adoption of the rule $a^{z+w}=a^za^w$ indicated that he was attempting to coordinate the processes of exponentiation with real power and with complex powers into a single process.

Thomas: When tasked with calculating $3i^{i+1}$, Thomas first wrote the definition $e^{(i+1)Log(3i)}$ and replaced Log(3i) with the definition, giving $e^{(i+1)(ln|3i|+i\cdot Arg(3i))}$, then replaced the magnitude and argument functions with their outputs, giving a final expression of $e^{(i+1)(ln(3)+i\frac{\pi}{2})}$, saying "honestly, I would leave it at just that." I asked if there was an alternate way to write it. Thomas proceeded by using the conjectured rule that $(ab)^z=a^zb^z$, transforming the calculation to $3^{i+1}\cdot i^{i+1}$ He wrote both factors in terms of the definition, similar to how he did with his first calculation. Because Arg(3)=ln(|i|)=0, he got the expression $e^{(i+1)ln(3)}e^{(i+1)(i\frac{\pi}{2})}$. Next, he distributed and used an exponent rule simultaneously to get the expression, $e^{iln(3)}e^{ln(3)}e^{-\frac{\pi}{2}}e^{i\frac{\pi}{2}}$ identifying the outermost terms as rotations and the inner terms as scalars. He then identified that $e^{i\frac{\pi}{2}}$ is a 90-degree rotation, and is therefore equal to i, and substituted, giving a final answer of $ie^{iln(3)+ln(3)-\frac{\pi}{2}}$. The final question I asked about this calculation was if it said anything about the final angle and magnitude. He "could simplify a bit more" using the inverse relationship between ln(x) and the exponential function, resulting in the expression "3ie$-\frac{\pi}{2}3^i$ but "in terms of angles I can't say much," and indicated that the magnitude could be read as a product of "all the scalars here," referring to 3i, $e-\frac{\pi}{2}$, and 3^i. There was no clear evidence if he thought of 3^i as a scalar, or just the 3.

When calculating $(-(1+i)^{-5-3i})$, Thomas again used the definition of z^w, writing $e^{(-5-3i)Log(-1-i),}$ and then the definition of the logarithm, writing $e^{(-5-3i)(ln|-1-i|+i\cdot Arg(-1-i))}$. Although he instantly calculated |-1-i| as $\sqrt{2}$, he drew out the argument to get $\frac{3\pi}{4}$, "because φ is between -π and π." He again did a multiplication and exponent rule simultaneously to get $e^{-5ln\sqrt{2}}e^{-i\frac{15\pi}{4}}e^{-3iln\sqrt{2}}e^{\frac{9\pi}{4}}$. Finally, using the inverse relationship between the exponential and logarithm, he got a final answer of $(\sqrt{2}\ \ )^5(\sqrt{2}+\sqrt{2}i)(\sqrt{2})^{3i}(\sqrt{2}+\sqrt{2}i)$, and commented that he would not know how to simplify $(\sqrt{2})^{3i}$, and that the furthest he could go in simplifying would be to identify the real and complex parts.

The simultaneity of operations indicates a process understanding of finding the magnitude and angle of a complex number, although these two processes did not appear to be coordinated into a process for calculating logarithms. Thomas also demonstrated a process understanding of complex multiplication but continued to rely on an action understanding of taking complex powers via the constant execution of a standard algorithm.

Comparing Magnitudes of Exponential Expressions

Question 1: per the framework described by Danenhower (2000). By this, I mean that both Thomas and Monk identified that the polar form of a complex number has an independent magnitude and angle component and that for the question, they only needed to isolate the magnitude. This is the "free modification" that is characteristic of an object-level understanding. Consistent with prior data, they both treated integer exponents as iterated multiplication. I infer this because they both alluded to notions of stretching and rotating and did not use the general definition of complex exponentials in their calculations. According to Thomas "the $e^{i\theta}$, that's always just going to be 1," and according to Monk, "This (referring to terms of the form $e^{i\theta}$) is just going to be our spinny component, so it'll just go away." They reasoned that because the base had a magnitude of 2, taking higher integer powers caused the sequence to increase.

Question 2: The second question caused the participants to think aloud the most out of the three as they used the techniques they developed for Question 2 for Question 3. Thomas used his prior observed method for computing $(-1+i)^{3+i}$, concluding that "the things that affect our length" were the terms $e^{3\ln(\sqrt{2})}e^{\frac{-3\pi}{4}}$, upon which he concluded that the sequence was increasing. Upon further questioning, he went back and forth between each factor determining the magnitude or only one. Initially, he thought that the increase of the real component of the exponent affected only the first term, but likely due to both factors having a 3, he changed his process to have each 3 increment by 1, resulting in this chain of expressions: $e^{3\ln(\sqrt{2})}e^{\frac{-3\pi}{4}}$,$e^{4\ln(\sqrt{2})}e^{-\pi}$,$e^{5\ln(\sqrt{2})}e^{\frac{-5\pi}{4}}$. However, he eventually realized that the second term of each expression corresponded to the angle of the base, which stayed constant, and was able to settle on a conclusion that the sequence was increasing. Notably, despite the length of the interview and his continued practice in calculating logarithms, he was still careful to write out the definition of logarithms to calculate his final answer. From this, it is evident that his thinking at that point relied on a mix of actions and processes, as he consistently deferred to the definition of logarithms, and kept track of each quantity so he would not have to redo the other calculations.

Monk operated with a different understanding of Question 2. He explicitly used the assumption that $z^{a+b}=z^a z^b$, and likely implicitly used the assumption that $|zw|=|z||w|$. Hence his sole focus on the term $(-1+i)^3$, ignoring $(-1+i)^i$, possibly because it was a common factor in all three terms. After calculating $(-1+i)^3$ as 2(i+1), he reasoned that higher powers would increase the magnitude, verbalizing the magnitude of each, concluding that the sequence is increasing. Essentially, he identified the consequential calculations to attend to and reduced it to a question about iterated multiplication of real numbers.

Question 3: Thomas approached Question 3 with his usual approach of decomposing each operation into its definition. However, once he arrived at the term $e^{(3+i)(ln|-1+i|+i\frac{3\pi}{4})}$, he replaced the (3+i) term with a variable, (3+ai), "since we're changing the coefficient in front of the i." He proceeded to distribute and use a power law to get $e^{3ln\sqrt{2}} \cdot e^{i\frac{9\pi}{4}} \cdot e^{ai\cdot ln\sqrt{2}} \cdot e^{\frac{-3\cdot a\cdot\pi}{4}}$. After simplifying each term, he identified that "the main thing here that's changing is this $\frac{3\cdot a\cdot\pi}{4}$." Although the third term also has an a inside of it, he reasoned that "this one's also going to be an angle, and we're just dealing with the modulus here… we know that we can split the modulus up across products, so $[e^{a\cdot i\cdot ln\sqrt{2}}]$ is equivalent to 1." He also noted that $e^{i\frac{9\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ "because of periodicity, we can subtract 2π from it." He concluded that because "our thing that's changing is $e^{\frac{-3\cdot a\cdot \pi}{4}}$, so a larger a means our overall quantity is decreasing" and concluded that the sequence is decreasing. There is evidence here that Thomas is operating with an object-level understanding of complex exponentiation, given that he established a useful form of the numbers he was working with and used a variable to reason about the problem. He developed an algorithm for this question and realized exponentiation as a function, corresponding to object-level understanding in Danenhower's (2000) and Pitta-Pantazi et al.'s (2007) frameworks. The question possibly made him generalize his schema of functions to accommodate complex exponents, as he first led me to believe that he would calculate each term individually.

Monk's initial reaction was that "[$(-1+i)^3$] will give us $2\sqrt{2}$ and [i] is going to spin it each time." From this, I infer that he knew immediately that $|-1+i|= \sqrt{2}$, and that cubing it would give it a magnitude of $2\sqrt{2}$. I also infer that he was breaking up the calculation into $|(-1+i)^3|\cdot|(-1+i)^i|$ This is further evidenced by his next move where he looked at the second term, and wrote $2\sqrt{2}|(-1+i)^{i+i}|$. He then broke $(-1+i)^{i+i}$ into $(-1+i)^i\cdot(-1+i)^{i+i}$, after which he said "and that's just taking your vector and spinning it so the magnitude isn't going to change, so it's just going to square itself." In this utterance, he partly associates purely imaginary powers with rotations. Next, he said, "Well, this is going to be a squared magnitude," referring to his inscription of $(-1+i)^{i+i}(-1+i)^{i+i}$ "And because the magnitude of that is -1+i is $\sqrt{2}$ because that's not less than 1" he concluded that the sequence was increasing. However, the magnitude of $(-1+i)^i$ is approximately 0.09, which would instead lead him to conclude the sequence was decreasing.

I conclude that Monk demonstrated a process understanding of complex multiplication and complex exponents with integer coefficients, given his instant claim that the magnitude of $(-1+i)^i$ was greater than 1. This is possibly an instance of the T$\mathbb{R}$D$\mathbb{C}$ phenomenon as described by Hancock (2018) where he described participants "extending real intuition to the complex setting erroneously," (p. 392) the intuition being that a complex number with magnitude greater than 1 raised to a power with magnitude greater than or equal to 1 will have magnitude greater than 1.

Discussion and Conclusion

In extending research on student understanding of complex analysis, I replicated the finding of Shifting Representations as found by Danenhower (2000), Panaoura et al. (2006), Karakok et al. (2014), and Soto-Johnson & Troup (2014) in how the participants tackled their calculations of complex exponentials. My participants chose the representations to use for a given question based on either its initial representation, or its representation in the definition of the complex exponential. They rarely experimented to see if alternate representations would be useful.

Because they only used familiar rules after they established their plausibility, participants seemed hesitant to leverage their knowledge of real-valued exponents and logarithms, instead reasoning from the definition. Only toward the end were instances of T$\mathbb{R}$D$\mathbb{C}$ seen explicitly. It is possible that during the experiment, the schemas related to complex and real exponents were separate, and the participants were actively generalizing both into a more general schema. Additional analysis of their performance on other items will need to be done to make definite conclusions, but this works toward a more complete understanding of the T$\mathbb{R}$D$\mathbb{C}$ phenomenon.

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