Developing Understandings of Accumulation: A Preliminary Research Report

Allison Olshefke-Clark
University of Delaware

This preliminary research report examines how undergraduates develop quantitative meanings of accumulation from rate, a foundation for understanding the definite integral and the Fundamental Theorem of Calculus. This study is a pilot study for a larger project. It consists of a pre-/post-assessment and a teaching experiment. The task sequence provides opportunities for students to develop productive meanings for speed, distance, and their relationship as rate of change/accumulation. Preliminary analysis of the pre-assessment indicates that students struggle to conceptualize average and instantaneous speed, which suggests that the teaching experiment can be productive. In this report, I outline the theoretical framework, task sequence, preliminary findings, and ongoing work. I conclude with areas for audience discussion: Seeking feedback on analysis and ideas for expanding upon this pilot study.

Keywords: calculus, definite integral, accumulation, quantitative reasoning, student learning

The definite integral is a concept used for mathematical modeling in many disciplines including physics (e.g., Griffiths, 2013), economics and business (e.g., May & Bart, 2024), and biology (e.g., Garfinkel et al., 2017). Approaches to teaching the definite integral in high school and college calculus courses often promote a meaning for the integral as the area of a region bounded by a function's graph and the x-axis (Huffman Hayes, 2024). However, this meaning of the definite integral is idiosyncratic to the Cartesian coordinate system (e.g., Moore et al., 2014), and it is not productive for modeling all physical contexts (e.g., Sealey, 2006) or for making sense of all mathematical situations, such as backwards integration (Bajacharya et al., 2023).

Over the past decade, a small but growing body of literature has advanced the idea that students should be supported in developing quantitative meanings for the definite integral, i.e., focusing on the meanings of each component of the definite integral in a situation (Jones & Ely, 2023). In their review of literature, Jones and Ely (2023) outline two distinct approaches that researchers have used to develop students' quantitative meanings for the definite integral: Adding up the pieces (AUP) and accumulation from rate (AR). In brief, the difference between the AUP and AR approaches lies mainly in the relationships and quantities that compose the integrand. With an AUP approach, one conceives of the integrand as a multiplicative relationship that forms the target quantity when these "tiny bits" of the target quantity are summed (e.g., Oehrtman & Simmons, 2023; Stevens & Jones, 2023). In an AR approach, the relationship between the integrand and target quantity is explicitly covariational in that it considers how the changing rate of the target quantity affects the total accumulation of the target quantity (e.g., coordinating how total distance driven varies with instantaneous speed; Thompson et al., 2013).

Most of the recent research on students' meanings for the definite integral advances approaches that are compatible with AUP (Jones & Ely, 2023). This emphasis in the literature base may be because the AUP approach is seen as more intuitive and accessible for many calculus students. Intuitive because an AUP perspective can be more easily visualized for many of the bounded, physical contexts that students encounter in their college coursework. Accessible because students often enter calculus with limited understandings of rate (Frank & Thompson, 2021; Thompson 1994), which may make using the AR approach more challenging.

I argue, however, that the AR approach has benefits that outweigh its challenges. The AR approach is derived directly from the Fundamental Theorem of Calculus (FTC), which is the backbone of the discipline and is a major idea in many calculus curricula (Huffman Hayes, 2024; Lanius, 2024). Thompson (1994) emphasized the FTC as "the realization that the accumulation of a quantity and the rate of change of its accumulation are tightly related—is one of the intellectual hallmarks in the development of the calculus." (p. 8). Despite its importance, students have difficulty conceptualizing and operationalizing the FTC, and proponents of AR believe that this approach can coherently guide students to understanding it (Thompson et al., 2013). Moreover, I posit that the AR approach also prepares students for authentic intellectual work in many STEM fields, like engineering, as accumulation from rate is a crucial idea in approximation and predictive modeling (A. M. Clark, personal communication, April 10, 2024).

Although studies have attempted to use an AR approach with students (e.g., Thompson 1994), more research has focused on the way that students develop understandings of the definite integral across a unit via an AUP framework (e.g., Lehmann, 2024; Stevens & Jones, 2023). Thus, there is a need for further examining the viability of using an AR perspective to develop ideas about the definite integral and FTC. This study, which is a pilot study for my dissertation, focuses on investigating how students' thinking about accumulation of rate (as a precursor to the definite integral and FTC) develops over a series of tasks that advances the AR approach. In this preliminary research report I describe the task sequence and theoretical framework underlying it. I then report the results from the pre-assessment and discuss the ongoing teaching experiment. Finally, I conclude with planned next steps for this study and pose areas for audience discussion I anticipate engaging in during the session.

Methods

This pilot study is comprised of two parts: A whole-class pre-/post-assessment and a teaching experiment with a pair of students (Steffe & Thompson, 2000). At the time of submitting this report, the pre-assessment data has been collected and the teaching experiment is ongoing. The teaching experiment will continue across the Fall 2024 semester, and the post-assessment will be conducted in early December 2024. Preliminary analysis of all data will be completed and addressed in my presentation at the 2025 RUME conference. In this section, I describe the setting and participants involved in this pilot study. I also outline the task sequence and the theoretical framework that underlies them.

Theory and Tasks

Carlson et al. (2003) detailed a framework of the requisite mental actions needed to engage with the FTC (Table 1). The mental actions trace the development of an understanding of accumulation from rate that progresses from discrete to infinitesimal intervals. Guided by this framework, I compiled a task sequence that I hypothesize will provide students structured opportunities to develop each of these mental actions.

Table 1. Carlson et al.'s (2003) mental actions of the FTC

MA1	Coordinating the accumulation of discrete changes in a function's input variable with the accumulation of the average rate-of-change of the function on fixed intervals of the function's domain.
MA2	Coordinating the accumulation of smaller and smaller intervals of a function's input variable with the accumulation of the average rate-of-change on each interval.

MA3	Coordinating the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change of the function from some fixed starting value to some specified value.

The pre- and post-assessment items are the *Round-Trip* task and the *Where's the Car?* task (Figure 1), which have the potential to elicit students' thinking about average rate of change and its accumulation in ways compatible with Carlson et al.'s (2003) framework. The purpose of the Round-Trip task (a version of San Diego to El Centro using relevant cities, which are masked for confidentiality; Hackworth, 1994) is to elicit teachers' meanings for average rate of change. It asks:

> A car went from College Town to The Beach, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to College Town if it were to have an average speed of 60 miles per hour over the whole trip?

The productive meaning for average speed is "the constant speed that the car would need to travel to go the same distance in the same amount of time as the actual trip," though teachers (and likely their students) often use the arithmetic mean to calculate the average speed (Yoon et al., 2015, p. 341).

The *Where's the Car?* task builds on this idea and is intended to provide students an opportunity to engage in MA1 (Table 1). In this task, students must use information about a car's instantaneous speed, provided in a table, to approximate an average rate of change for each discrete interval and determine the accumulated distance. The prompt and corresponding table for this task is shown in Figure 1 below.

The table below shows information about a car's speed at various time after the engine started. Assume that the car travels in a straight line after the engine starts. Approximately how far does the car travel in these 30 seconds?

Time since the engine started (seconds)	**Speed** (meters/second)
0	0
5	11
14	20
30	25

Figure 1. The Where's the Car? task

The teaching experiment consists of a series of four additional tasks. The first task, *Over and Back* (Thompson & Thompson, 1994), is used to further determine students' understanding of speed and develop productive meanings for average and instantaneous speed. Students are presented with an interactive applet of a 200ft race between a rabbit and a turtle. Students can change the animals' speeds to answer a variety of questions designed to promote an understanding of speed as the rate of change of distance (e.g., "Give Rabbit a speed that will make him go over and back in 7 minutes"). Once students have developed meanings for speed conducive to engaging with its accumulation, students will then have an opportunity to develop MA1 by engaging with the *Which Racetrack?* task. This is an extended version of *Where's the Car*? in which students use given speed data to determine which racetrack a car was driving based on its total distance. In *Designing a GPS,* students build to MA2 by gathering instantaneous speed information from a video of a racecar's dashboard to track the ongoing distance accumulated. By having access to a relatively continuous stream of instantaneous speed data, I will challenge students to "coordinate the accumulation of smaller and smaller intervals" (Carlson et al., 2003, p. 167) in order to calculate better estimates of the distance and begin to explore the idea of instantaneous speed. The final task, *Drag Race,* builds to MA3. This task asks students to use a model of a racecar's speed in a drag race expressed in function notation, i.e., $v(t) = 71 \times (1 - (1/3)^t)$, to determine the car's distance from the starting line (i.e., distance accumulated) at specific times and eventually at any time. By using an explicit function, students can progress from MA2 to MA3 by considering the "instantaneous rate-of-change of the function from some fixed starting value to some specified value" (Carlson et al., 2003, p. 167).

Participants and Analysis

The participants in this study are a group of 12 undergraduate students enrolled in an undergraduate mathematics course focused on covariational and quantitative reasoning at a large, public university in the Mid-Atlantic United States. All students have previously taken Calculus I and thus have been introduced to the integral and FTC, though it is unknown to what extent they have had opportunities to think about these topics quantitatively. The whole group will take the pre- and post-assessments, and a pair of these students are the focal students of a teaching experiment. As a teacher-researcher, I can engage deeply with the thinking of the focal pair of students. In my instructional role for this course, I will also be aware of the progress of the non-focal students over the semester. This dual position will allow me additional insight as I analyze the changes between the pre- and post-assessments and conduct analysis.

During the first course session, the instructor administered the pre-assessment, which was composed of two items designed to probe students' incoming meanings for average rate of change and its accumulation. Students completed the pre-assessment independently. I coded student responses based on my interpretation of the type of quantitative meaning(s) for speed that the student used to solve the problem. The coding scheme for the *Round Trip* task was influenced by the rubric developed for the *San Diego to El Centro* task (Yoon et al., 2015) but was adapted to more closely characterize the meanings exhibited by this group of students. The coding scheme for *Where's the Car?* was developed inductively (Saldaña, 2021) and is presented in the following section.

Preliminary Results and Data in Progress

In this section, I present preliminary results based on students' responses to the two pre-assessment tasks. I also briefly discuss anticipated results from the ongoing teaching experiment.

Pre-Assessment Results

Most students did not exhibit evidence of quantitative meanings for speed that allowed them to productively calculate and/or accumulate average speed in either pre-assessment item.

In the *Round Trip* task, only four (33%) of the responses indicated a final answer that was a value for speed. Of these, only one response correctly identified the necessary return speed as 120 mph. The remaining three responses used the concept of arithmetic mean to calculate the needed return speed as 80 mph, echoing Yoon et al.'s (2015) findings. The final eight (67%) of the responses did not address the written prompt. These responses either did not clearly indicate a final answer (n = 3) or provided a final answer that was not a speed (n = 5). For instance, three students indicated their final answer was "1.5 hours." Notably, this is an intermediate measurement towards calculating the correct answer: 1.5 hours is the time it takes to travel 90 miles at 60 miles per hour.

Responses to the *Where's the Car?* task followed a similar breakdown. Three (25%) of the responses are potential evidence of MA1. These responses used the instantaneous speeds indicated to calculate a distance the car traveled over intervals based on the information in the table and then summed these distances to find the total distance over 30 seconds. A further four responses (33%) presented intermediate calculations that were consistent with MA1 activity within a single time interval, but did not calculate or accumulate the subsequent distances. For instance, some calculated that in the first five seconds the car traveled 55 meters, but they did not calculate the distance traveled in any other interval. Four students (33%) did not exhibit productive quantitative meanings for speed. The most common responses in this category added up the four provided instantaneous speeds and indicated the resulting sum as the total distance traveled. A final student simply responded, "I'm not sure."

Data in Progress/Anticipated Results

The results from the pre-assessment suggest that the students will benefit from further opportunities to develop quantitative meanings for speed (average and instantaneous) and distance as well as the relationships between them. I intend the ongoing teaching experiment to provide these opportunities, and I anticipate that engagement with the task sequence described above will help students develop productive meanings for these quantities in relation to the definite integral.

I have selected two students who did not get either pre-assessment task correct to serve as my focal pair for the teaching experiment. Through working with these students on the task sequence over two course sessions (about six total hours) I anticipate that I will be able to model how their thinking about accumulation from rate of change develops. In particular, I predict that by the end of the teaching experiment I will have clear evidence of the students operating with each of Carlson et al.'s (2003) mental actions.

Potential Areas of Audience Discussion

By the time of the conference in February 2025, I will have collected, compiled, and begun to analyze all the data from this study holistically. This includes the pre-assessment, exploratory teaching, and post-assessment data. Because this is a pilot study for a larger project, getting feedback from an audience at this stage of the research process will be incredibly helpful in refining the findings for presentation to my committee and in brainstorming how this work can be expanded upon for a larger study. I also hope that other researchers who investigate students' thinking about the definite integral will share their reactions to my findings, especially those who focus on quantitative approaches. Additionally, I am interested in further examining the benefits and drawbacks of the AUP and AR frameworks in different situations. Specific questions I intend to ask the audience include:

1. How does the analysis I present compellingly show students' developing reasoning within the AR conception of the definite integral? In what areas could I provide more nuance or further evidence?
2. In what ways do you think this study could be expanded upon? Are there any lines of thinking that would be particularly interesting to follow? Any new areas that would be useful to explore?
3. For calculus instructors: In what ways do my findings resonate with or challenge your experiences teaching the definite integral to undergraduates? What implications might these findings have for your own practice?

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