Student Challenges in Proof by Mathematical Induction: A Preliminary Analysis of a Generalizing Study

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Many studies have investigated the challenges students face during proof by mathematical induction (PMI). However, the majority of these studies are drawn from a small sample size. In order to generalize the findings from previous literature, we draw from a study of over 1000 proofs to develop an analytical framework for identifying common student challenges in PMI. In this preliminary analysis, we report our finding from analyzing 180 written responses for three types of PMI tasks.

Keywords: Proof by Mathematical Induction, Student Challenges

It is well documented in the literature that students often find proof by mathematical induction (PMI) particularly difficult (Styliandines et al, 2007; Norton and Arnold 2019; Baker 1996). The majority of these studies have used qualitative techniques to analyze students' written proofs, student interviews, and/or classroom recordings and observations from a small sample of data. However, to advance the field forward in terms of the generalizability of previous findings (Melhuish and Thanheiser, 2018), a study on student challenges in PMI using a larger data sample is yet to be conducted. In light of this, in this preliminary study, we document our efforts in analyzing student challenges of PMI by analyzing a subset of data from a larger sample. We first discuss existing findings on student challenges in PMI, report our study where we analyzed 180 out of 1019 proofs, and finally discuss implications of our findings and our future steps.

Empirical Background

Many scholars have investigated students' cognitive gaps surrounding proof by mathematical induction using a variety of methods (e.g., Dubinsky, 1986; Styliandines et al., 2007). For example, in a study with 40 high school and 13 college students, Baker (1996) reported that students displayed a variety of challenges including in mathematical content and deductive knowledge. For instance, Baker (1996) reported that students had difficulty in connecting the significance of proving the base case with the formulation of the inductive hypothesis, students applied the inductive step to all cases of the inductive statement, and students relied mostly on procedural techniques over the conceptual understanding of the proof strategy. Through a classroom-based study with pre-service teachers, Davis et al (2009) found 4 main challenges students experienced during PMI: problem-solving, connecting information from multiple areas, validating others' proofs, and infinite iteration. In a different study, through analyzing 67 student responses and interviewing 8 of those students, Taitra (2023) found that while most students in the study were able to set up the inductive step, the students found proving the inductive step to be challenging. Reid's (1992) study on students' understanding of proof by induction revealed a key finding: students struggled to connect induction with recursive thinking. Studies have also shown that students often take a results-oriented approach to writing induction proof (Harel, 2001). That is, students may use examples to verify the truth of statements (Baker, 1996; Harel, 2001), use a backward strategy while writing proofs (Baker, 1996), or treat proof by induction as a list of steps that need to be checked (Harel, 2001; Styliandes et al, 2007; Knuth, 2002). Building on our prior work (Authors, Year), we intend to contribute to and extend the research on student challenges in PMI by analyzing 1019 student responses to proof by inductions surrounding divisibility, summation, and recursive relations. In this preliminary study, we address the following research question: *What challenges did students face in PMI as identified from our analysis of 180 written responses across three types of PMI tasks?*

Methods

Data for this study comes from a larger project, where the project goal was to measure the effectiveness of completing Proof Blocks, a software that enables students to construct proof by arranging pre-written statements, over reading a book chapter (see Poulsen et al., 2024). As part of the larger project, in total we collected 1019 written responses for the proofs shown in Figure 1 from students enrolled in a discrete mathematics course (The full data set can be found in Poulsen, 2024).

Divisibility: For any natural number n, $2n^3 + 3n^2 + n$ is divisible by 6.
Recurrence: Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by • $g(0) = 0$ • $g(1) = 4/3$ • $g(n) = 4/3\, g(n-1) - 1/3\, g(n-2)$, for $n \geq 2$. Use induction to prove that $g(n) = 2 - 2/3^n$ for any natural number n.
Sum: Claim: $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for all natural numbers n.

Figure 1. PMI tasks

In this preliminary analysis, we analyzed 180 proofs (60 divisibility, 60 summation and 60 recursive) with the aim to develop an analytical framework that can be applied to the entire data set. To identify student challenges in proof by mathematical induction, we initially coded our data using an *a priori* framework derived from the literature. Four research team members independently applied these codes to sections of the data and convened weekly to address any discrepancies. Resolving these disagreements often resulted in adjustments to the existing codes or the development of new ones. Adjustment to existing codes took the form of expanding the definition of each code, condensing multiple codes into one. When aspects of student work could not be coded by existing code, a new code was created and defined. The new codes and modification of existing codes were checked against the already coded data set to ensure consistency in our use of the codes. In the following section, we present the codes as findings of student challenges in PMI that we have developed up to this point of the study.

Findings

We present the overview of the themes, the corresponding sub codes, and their definitions that emerged during our analysis in Table 1. Due to space constraints, we illustrate four of the sub-codes.

Table 1. Label

Major Themes	Sub-codes	Definition
Quantification Related challenges	The induction variable, k, is used without quantification	Proof uses a variable that represents the specific value the inductive hypothesis (IH) to be true up to, without defining that variable beforehand

		(e.g. indicators include the proof not having explicit language such as "for some k..." or "there exists k...")
	Conflating between the roles of n and k.	Uses the same variable in the inductive hypothesis to represent an arbitrary value and a specific value.
IH Related Errors	Assuming the IH true always (instead of for a particular value)	Assumes the IH without quantifying the variable or allows the variable to be any natural number. The proof must have some IH in order for this code to apply.
	Mismatch between values assumed in IH and value proved in inductive step (IS).	Value proved in IS is not the successor of the value assumed in the IH.
	IH is to be proven, not assumed	The proof shows no indication that the IH is something that is assumed, and instead claims that either have proved it or will prove it.
	Using IH inductive hypothesis without assuming it	No assumption is made for the statement being true. Proof jumps right into proving k+1, with no basis for k being true.
	Confuse recursive definition with IH	Proof assumes the definition of the recursive function rather than assuming that the recursive function is equal to the closed form equation for some k.
Base Case Related	More base cases than necessary	Proved for 2 or more base cases when only needed 1, or 3 or more when only needed 2
	Not enough base cases	Proved for 1 base case when they needed 2.
	Not proving the property is true for base cases of the recurrence relation	The proof does now show that the recurrence relation is equal to the closed form for base cases. In particular, there is no evidence shown in the proof that the closed form expression for 0 and 1 are equal to $g(0)$ and $g(1)$ respectively as defined for the recurrence relation.
Domain-specific knowledge	Showed knowledge of PMI outline, but not filled in	Proof outlines the complete steps of PMI, but didn't provide details.
	Developing an understanding of the summation sign	Incorrect use of the summation operator in the proof.
General Logic Errors	Treating an expression or function as a proposition	The proof treats an expression which does not have a truth value in place of a proposition which does have a truth value.
	Backwards proof	Starting with the equality to be proved and manipulating both sides (rather than starting

	from one expression, and manipulating it to show it equal to the other)
Incorrect lower bound for induction variable	The lower bound for the inductive variable doesn't allow the IH to be soundly applied to complete the proof.
Proof by example	No rigorous proof is given. Proof is instead given by a few examples, implying that proving few examples proves the entire statement.
Assumed only $g(k)$ when they also needed to assume $g(k-1)$	The proof only prioritizes one recursive relationship in the IH and does not assume the IH to be true for the previous two recursive relationships.

Implicit Quantification of the Induction Variable

In this vignette, we provide an example proof (Figure 2) where the induction variable k is used without quantifying it. Here, the student seems to understand that, in the IH, the equation holds true for all natural numbers up until an arbitrary value, (represented by the variable k), but the proof does not define this variable. While the student's intentions of using k in this context may not be difficult to determine, the proof shows no evidence that the student may have an understanding of the significance of defining k in PMI.

Proof: by induction on n

Base: Let n = 0. Then $\sum_{p=0}^{0}$(p * p!) = 0 an (0+1)! - 1 = 0.

Induction: Suppose $\sum_{p=0}^{n}$(p * p!) = (n+1)! - 1 for all n = 0, ..., k -1.

Then we must prove that $\sum_{p=0}^{k}$(p * p!) = (k+1)! - 1

Figure 2. Student proof for summation task where k is used without quantification

Incorrect Lower Bound for Induction variable

In this vignette, we provide an example proof (Figure 3) where an incorrect lower bound for the induction variable is used. In the proof in Figure 3, the base case is proven for when $n = 1$. Therefore, the induction variable k, should iterate between the 2nd case to the nth case. However, in the proof, the student sets the lower bound for k as n, in their IH statement.

We will use proof by induction.

Inductive Predicate: $\sum_{p=0}^{n}$ = (n + 1)! - 1, for all natural numbers n.

Base Case: Note that when n = 1, $\sum_{p=0}^{n}$ = (1 + 1) - 1 = 2.

Inductive Hypothesis: For some k, $\sum_{p=0}^{n}$ (n + 1)! - 1 for each n . . . k.

Inductive Step: For some k, $\sum_{p=0}^{n+1}$ (n + 2)! - 1 for each n . . . k - 1.

Figure 3.Student proof for summation task with incorrect lower bound for induction variable.

Mismatch Between Values Assumed in the Inductive Hypothesis and Values Provided in the Inductive Step

In this proof (Figure 4), the value shown in the inductive step is not the successor of the value assumed in the inductive hypothesis. The proof assumes that the inductive hypothesis holds true for values from 0 to k. However, while the goal is to prove the statement for $k + 1$, the proof only proceeds to prove it for k, as indicated by the "need to prove" statement.

next, suppose the hypothesis that for some k, $\sum_0^n$ (n * n!) = (n + 1) ! - 1 in k = 0, 1......k

we need to prove that $\sum_0^k$ (k * k!) = (k + 1) ! - 1

Figure 4. Student proof for summation task where there is a misalignment in values

Assuming the Inductive Hypothesis is True Always (Instead of for a Particular Value)

In this proof (as shown in Figure 5), the inductive hypothesis is assumed to work for all natural numbers. This assumption makes the proof concerning $n + 1$ obsolete, as $n + 1$ is an element of all natural numbers, which was already assumed.

Base case: n = 0. $\sum_{p=0}^{n}(p \times p!) = 0 = (n+1)! - 1$.

Inductive hypothesis: Assume that the claim works for all natural numbers n, and prove that it works for natural number $n + 1$.

Inductive steps: $\sum_{p=0}^{n+1}(p \times p!) = (n+1)(n+1)! \times \sum_{p=0}^{n}(p \times p!) = (n+1)(n+1)!((n+1)! - 1)$. Then, we can rewrite the function, so it will be $(n+2)! - 1$.

Figure 5. Student proof for summation task where inductive hypothesis is true always

Discussion and Conclusions

In this report, we presented our preliminary findings from analyzing 180 written proofs that were drawn from a larger data set comprising 1019 proofs. Our objective is to utilize the analytical framework presented here for the remaining proofs, while continuously refining and enhancing it to ensure the development of a comprehensive and rigorous analytical framework. Some of the findings reported here comport with previous findings from the literature. For example, Baker (1996) also found that his students used specific examples to complete PMI. Similarly, Stylianides et al. (2007) found that the pre-service teachers in their study relied on a list of steps to prove PMI. This, we believe is similar to what we identified as "showed knowledge of PMI outline, but not filled in," re-affirming students' procedural over conceptual understanding of PMI. Additionally, we also found some student challenges in PMI that were not reported elsewhere. For example, developing and understanding of the summation sum, mismatch between values assumed in IH and value IS, using IH inductive hypothesis without assuming it, and treating an expression or function as a proposition were unique to our data set. These newer themes may be attributed to our sample size, type of proofs (i.e., proofs that require summation operation and recurrence relations), and students' mathematical background. However, we acknowledge certain limitations to our findings. First, the data sources were only written proofs. These data sources are limited for uncovering the thinking behind students' mathematical decisions. Therefore, our claims may not accurately reflect students' understanding of PMI. Second, to determine whether or not a code/theme truly represents a common challenge faced by students during PMI, we need information about the prevalence of that code/theme. Our future study includes examining the frequency of all codes. Adopting an asset-based perspective (Adirejda, 2019), we echo other scholars (e.g., Norton et al., 2023) in emphasizing the need to explore the underlying reasons for students' difficulties and further advocate for research that examines the effective strategies instructors of PMI can employ to address these challenges.

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