Mathematicians Reject, Teachers Embrace: Different Perspectives on the Role and Significance of Disciplinary Obligations in Secondary Mathematics Teaching

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A key objective of academic mathematics courses in teacher education is to enhance teachers' awareness of disciplinary values and principles, ensuring alignment between mathematics education and the discipline. However, research indicates that this goal is often not met. A possible explanation is that mathematicians and teachers have different perspectives regarding the discipline's role in mathematics education. This study examines these perspectives in three extreme cases, where mathematicians and secondary mathematics teachers disagreed on the legitimacy of secondary mathematics activities from a disciplinary standpoint. Analysis of professional obligations that underlie the mathematicians' and teachers' reasoning reveals distinct interpretations of the obligation to the discipline and of its implications and significance in consideration of other obligations. These findings emphasize that teacher education should go beyond fostering awareness of disciplinary values and principles, and support teachers in learning how they can apply this understanding effectively in school to guide and inform their teaching.

Keywords: Teacher education, Mathematicians, Professional Obligations, Decision-Making

Introduction

One of the key roles of academic mathematics (AM) in the education and professional development of secondary mathematics (SM) teachers is to help teachers orient their teaching so that it aligns with and represents the values and principles of the discipline (Ball & Bass, 2009; CBMS, 2012; Cooper & Pinto, 2017; Hoffman & Even, 2023; Shulman, 1986). A common approach in AM courses for teachers is that mathematicians provide teachers with AM learning experiences, with the expectation that this experience will trickle down into their classroom instruction (e.g., Wu, 2011). However, current research and policy documents do not specify how this orientation is supposed to happen in practice, nor how AM learning experiences become a resource for teachers' instructional decisions. There is empirical evidence that both teachers and mathematicians do not always have a clear vision of how AM studied by teachers is brought into classroom teaching (e.g., Lai, 2019; Wasserman, 2018; Zazkis & Leikin, 2010).

Therefore, in recent years, there has been a growing interest in the subtle process by which aspects of AM may become a resource for SM teachers' instructional decision making (e.g., Biza et al., 2022; Herbst & Chazan, 2020; Pinto & Cooper, 2022, 2023; Zazkis, 2020;). One approach for delineating these processes is by exposing how the obligation to the discipline may orient teachers' decision making (Herbst & Chazan, 2020). Specifically, Herbst and Chazan (2020) propose that mathematicians will share their mathematical knowledge and practices with teachers to develop their sense of the discipline and inform their teaching. However, Herbst and Chazan also emphasize the crucial role of SM teachers in revising the initial ideas offered by mathematicians in consideration of the realities of SM education as manifested in other professional obligations. In this study, we adopt and adapt this approach by engaging mathematicians and experienced SM teachers with authentic SM instructional situations in ways that invite both sides to interpret and negotiate the role of obligation to discipline with respect to specific and concrete teaching decisions.

This paper is focused on three extreme cases where mathematicians deemed mathematical activities as inappropriate for secondary education since they breach values and principles of the discipline. The SM teachers, while acknowledging the breach, still considered using these activities in their classes. In these contexts, we investigate differences between mathematicians' and teachers' perspective regarding how obligation to the discipline should orient SM teaching. Our investigation is guided by the following research question: *How do mathematicians and SM teachers perceive the role and significance of the obligation to the discipline in relation to concrete instructional situations in SM education?*

Theoretical Framework

Teachers and mathematicians may be considered as different communities of practice. According to Akkerman and Bakker (2011), encounters between such communities, in which issues related to the practices of each community are discussed, may reveal a boundary – 'a sociocultural difference leading to discontinuity in action or interaction' (p. 133). For example, differences in professional obligations expressed by members of these communities reflect such discontinuities. The cross-community dialogue in these meetings provides a fertile ground for initially identifying these differences, and negotiating the role and significance assigned to them.

Herbst and Chazan (2020) identified four central professional obligations that guide mathematics teachers in making instructional decisions: the obligation to the discipline, the student as an individual, the class as a group, and the school. In articulating these obligations, they point out that mathematics teaching is expected to be influenced by the discipline. At the same time, they recognize that instruction takes place within a specific context, involving the student, the classroom, and the institution. These obligations reflect the tensions underlying teachers' decision-making. In extreme cases, tensions sometimes turn into an outright contradiction between obligations. Such contradictions reveal different perspectives on the role and significance of the obligation to the discipline in consideration of the context, and how this obligation may orient the teaching.

Methods

This research is part of a long-term research project – M-Cubed (Mathematicians, Mathematics teachers, Mathematics), conducted in Israel between 2019-2024 (Goor et al., 2022; Pinto & cooper, 2022, 2023). In this project, small groups of mathematicians and experienced secondary mathematics teachers watch videotaped lessons together, collaboratively exploring the mathematical and pedagogical issues they identify therein. The design and facilitation of M-Cubed is informed by the literature on boundary-encounters and boundary-crossing (Akkerman & Bakker, 2011). The videotaped lessons are carefully selected by the mathematics education researchers who lead the meetings, aiming to reveal professional dilemmas arising from the exploration of practices derived from AM, as well as the pedagogical vision of both SM teachers and mathematicians. Thus, this research setting serves as a laboratory for generating and analyzing the implications of ideas and perspectives from academic mathematics in the context of real-world instructional situations. Data for this study include video recordings of approximately 30 meetings, each with a duration of about three hours. A summary was prepared for each meeting, and selected excerpts were transcribed according to the research goals.

This study focuses on extreme cases in which mathematicians strongly opposed the teaching activities presented in videotaped SM lessons, deeming them mathematically flawed and inappropriate for secondary education. In contrast, teachers did not accept the mathematicians' critiques at face value and advocated for using these activities despite their apparent misalignment with the discipline. These cases were selected to expose differing perspectives on the role and significance of disciplinary values and principles in the context of authentic SM teaching decisions. The contrasting views between the mathematicians and the teachers prompted both groups to articulate their tacit perspectives, facilitating the identification and characterization of various professional obligations that guide and constraint teaching decisions. To this end, we first analyzed the reasoning of both groups regarding the legitimacy of the activities, identifying how they referenced broader principles or values that extend beyond the specific cases yet impact decisions about using such activities in SM lessons. We characterized these principles and values according to Herbst and Chazan's (2020) four professional obligations, guided by descriptions and examples of these obligations in the literature. We then compared the obligations to the discipline expressed by mathematicians and teachers, and how these shaped their reasoning and conclusions. Additionally, we explored how they positioned the obligation to the discipline with respect to other professional obligations. In this paper we present an analysis of three extreme cases that illustrate how mathematicians and teachers perceive differently the role of disciplinary values in SM teaching, and how obligation to discipline may intersect with other professional obligations.

Findings

In each of the three cases we will describe, we first present the mathematical and pedagogical context under discussion. We then characterize the mathematician's obligation to the discipline as reflected in their critique of the activity. Finally, we describe the teachers' voices that emerged throughout the discussion, emphasizing the different perspectives of the teachers on the roles and significance of the obligation to the discipline in consideration of other professional obligations.

Case 1: Mathematical Statements Teachers Make Need to Be True.

This case revolves around a task from an 11th grade advance-track class that the teacher, Sergei, introduced as an example that underlines the need for intuitions in mathematics. The task is to determine whether a plane 'like' the plane illustrated in Figure 1a exit, given that $EK \nparallel AB$ and $FG \nparallel AB$. In the lesson, Sergei argued that there cannot be a plane that contains the points E, F, G and K, as such a plane would necessarily also contain the segment AB and therefore will necessarily contain the face ABD, which 'clearly' cannot be.

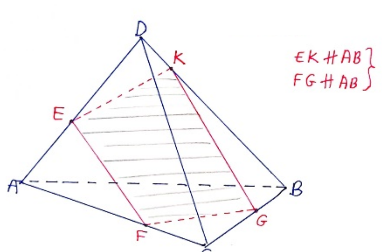

Figure 1a. Reconstruction of Sergei's board

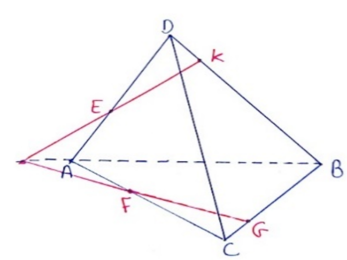

Figure 1b. Reconstruction of David's board

One of the Mathematicians, David, immediately recognized that in contrast to Sergei's argument, given a tetrahedron with vertices A, B, C and D, there always exists a plane that intersects BD, AD, BC and AC at points K, E, G and F respectively such that $EK \nparallel AB$ and $FG \nparallel AB$ (see Figure 1b). He further noted that this plane would not look 'like' the plane in the drawing on the board, but that this drawing cannot be taken as data. He contended that Sergei made a very critical error and framed the lesson as a "mathematical catastrophe":

> [Sergei] gave the [students] a statement that is not well defined, with conditions whose sole purpose are to confuse the students because they are not relevant. False data, a false statement, a false proof, what could be worse in a mathematics lesson?

David's criticism of the task is clearly in disciplinary values, as he makes no reference to the individual student, to the class, or to the school. Notably, David recognized potential pedagogical value in the task. When considering teaching alternatives, he suggested that after convincing students that the proof is correct, a teacher may lead students to explore and where the argument fails, which in his words would afford students an "amazing lesson." However, David insisted that students cannot be left believing a false mathematical claim that a teacher made.

Some of the teachers disagreed with this position in ways that suggested that the teachers were oriented by additional obligations that were not outweighed by their obligation to the discipline. For example, Vera, noted that she loved what Sergei did in the videotaped lesson, and would use this task in the same way. She noted that as a teacher, the issue that David highlighted does not bother her as much:

> OK, I'm convinced [there exists a plane]. But, if from now on I will be afraid to bring [to class] something beautiful because maybe somewhere, mathematically, I'm wrong, then I will never bring anything. Because there is a difference […] between a teacher that wants to show the beauty, that wants to light up [students' faces], that wants to get them to think, and between the mathematician that must have everything precise, precise, precise.

In contrast to David's criticism of the task, Vera centralizes the students when considering the use of the task in class. She considers mathematical errors an acceptable price she is willing to pay to be able to share with her students' beautiful pieces of mathematics. She considers the difference between her positions and David's as representing a distinction between how teachers' and mathematicians' disciplinary obligation to precision.

Case 2: The Mathematics Discussed in Lesson Needs to Be Well-Defined

This case revolves around the probability game introduced by a teacher, Ari, in his 8th-grade advanced class. In the game, called 'Rive-Crossing Game' (RCG), students are given a board as described in Figure 2a, and 18 office clips, which they place on numbered squares ranging from 2 to 12. Players take turns rolling two dice, and if the sum of the dice matches a number on which a clip is placed, the player can move the clip across the "river". The objective is to be the first to transfer all clips. During the observed lesson, Ari guided the students in constructing a graph that represents the optimal clip distribution on the board (see Figure 2b).

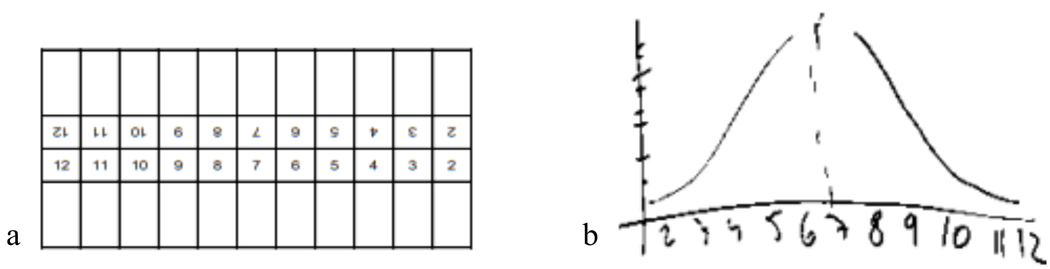

Figure 2. (a)The RCG's board and (b) A snapshot of board with the graph Alex built with his students

One of the mathematicians, John, identified the game as complex to solve, in terms of the mathematics underlying the game. He suggested the solution was likely not optimal: "I don't understand, what is the optimal strategy… I think that without a computer I don't stand a chance of calculating an optimal strategy." And then he explained: "… It's possible that there is a strategy with a slightly better expectation and slightly worse variance and another strategy that is the other way around. It's a very sensitive game."

In light of this perspective, during the continued discussion, John and another mathematician, Bill, explained they couldn't bring a mathematical problem to the classroom without being able to provide intuition for it, due to the complex mathematics behind it.

> Bill: Because I don't understand it myself. It's really hard for me to bring in something, where even on an intuitive level, I'm not entirely sure what's happening. Okay? Because I agree with John, that when the numbers are very large, it's better to approach this curve… As long as every throw moves a clip, there's no problem. The problem starts when we begin hitting zero, right? And that's something that will happen in smaller trials. It's a problem when I can't provide intuition myself.

In this case, as well, the mathematicians' critique of the task did not address the students, the class, or the school context. Again, it was grounded in disciplinary values. Specifically, the obligation to fully understand the mathematics brought to the classroom and the ability to model the problem, and well define it, to analyze it. During the discussion, John noted that without fulfilling this obligation, the RCG game loses relevance to probability learning, saying: "It is strategy thinking, game thinking. Is it related to probability? So-so".

Some of the teachers disagreed with the mathematicians' view on the activity's legitimacy. They didn't dismiss the claim that the task compromised the obligation to the discipline but gave it less weight compared to other interpretations of the disciplinary obligation the task invited, as well as other professional obligations. For instance, one of the teachers, Leah, argued that the stage of choosing how to distribute clips at the start of the game fosters initial intuition, mathematical thinking, and discussion. Unlike the mathematicians, she placed less importance on justifying optimal strategies through detailed mathematical analysis, due to time constraints that prevent students from reaching such advanced questions. Additionally, she highlighted the game's value in terms of her obligations to her students and the class, given the enjoyment it provided. She weighed her arguments against the 'damage' pointe out by the mathematician, comparing it to the 'benefits' she pointed out, as she said:

> … I don't understand what the big problem is. I see a lot of benefits. ...What's so scary … it's exactly what he was trying to say—frequency versus probability… as they said here, there's not enough time to play. They won't see the optimal strategy

And she added:

> I just want to defend the game as well. ... With all the disadvantages you listed. first of all, it's really fun. It's much less fun to flip a coin 100 times. Sorry, but that's super boring… It [RCG] really sparks a lot of interest. Kids talk about it. …usually, it works out for the ones who don't ignore the smaller tiles, who actually spread things out like a bell curve... And it's also really fun when you think you've figured out the optimal strategy, but you don't win... I think John said… opening with games. And especially with a game like this, it totally fits that purpose. In my opinion, it's a really good game.

Case 3: Motivation to learn a new mathematical idea needs to be authentic.

This case revolves around an activity from an 'introduction to function analysis' lesson in a 10th-grade high-track class. As part of the task, a student volunteer leaves the classroom while the teacher momentarily displays on the board a graph of an unfamiliar function, such as the graph of the function $f(x) = x^2(x + 3)$. Upon the student's return, the class was asked to verbally explain how to draw the graph, without using gestures or images. One of the mathematicians, John, was highly critical of the task from the start, questioning whether it is even mathematical. He explained: "You just start with a figure and end with a figure, so [it's] more a discussion in visual communication, not in mathematics". He elaborated on what he would expect from a task designed to motivate function analysis, clarifying what he meant by "not mathematics":

John: It all depends on the selection of a function, right? If you choose a function that looks like the back of a crocodile, they [students] will say it looks like the back of a crocodile, and there would be no mathematical at all

Facilitator: …But if you wanted to use mathematical concepts to describe a crocodile's back… you could say the function has many local maxima, here it's convex…

John: It wouldn't be natural. Anyone who would work this way would be using mathematics not for its purpose, in a way, right? We want mathematics to be used in places where it is natural… and so the function needs to invite a mathematical description.

John went as far as describing the lesson as 'terrible' and some of the other mathematicians supported his stance. In their critique, they generally did not consider the specific teaching context—the particular classroom, the specific students, or the institution. Instead, their concern indicated a strong obligation to the authentic way of things to be done in the discipline – arising naturally from a genuine need for mathematics, such as modeling in physics, rather than creating an artificial use of mathematics.

The mathematicians' utter dismissal of the task led one of the teachers, Emily, to press John to explain why he found the lesson so "terrible". Additionally, she elaborated the goals of the whole lesson and particularly the value of the activity from her perspective. In her explanation, Emily described a teaching process intended to create a need for a shared language in order to analyze functions, as another teacher explained it before John's critique:

> The goals of this lesson were different, in my view… they've spent a whole year learning about straight lines, and another whole year on parabolas. so, what now? Are we going to spend a whole year on cubic functions? … The idea is that we need tools to help us investigate functions in general... and regarding the second task, drawing something you haven't seen before. That's really... the motivation behind it is to develop a common language. Using it is part of the motivation.

Emily further elaborated: "All the words we use now, such as increasing, decreasing, positive, negative […] part of important concepts, [they] still do not have this language."

Emily and other teachers envisioned a process that achieves a pedagogical goal—creating a need for new mathematical ideas among the students. Her entire description was based on mathematics, but it is possible that her interpretation of mathematical motivation was more focused on school mathematics, whereas the mathematicians perceived it in its broader sense. Additionally, during the discussion, teachers also mentioned the students' enjoyment of the activity, which provided additional value to the activity.

Discussion

This study examined how mathematicians and SM teachers relate to breaches of disciplinary values and principles of the discipline in three extreme cases where mathematicians and teachers disagreed about whether mathematical activities are suitable for SM education. Analysis of the professional obligations that emerged in the mathematicians' and teachers' discussions of these activities highlights differing perspectives regarding the role and significance of obligation to the discipline in SM teaching. Our findings illustrate why awareness of disciplinary values and principles that teachers are expected to develop during their academic studies does not always manifest in their teaching the way mathematicians might expect. While this phenomenon has already been discussed in the literature (Herbst and Chazan, 2020), the cases in our study stand out since they illustrate vastly different perspective on obligation to the discipline even in cases that from a disciplinary perspective could be considered as 'black and white', and even when mathematicians have the opportunity to explain their perspective and persuade the teachers.

In all three cases, mathematicians identified the activities as breaching their obligation to the discipline so fundamentally that they rejected their suitability for SM education. The teachers, however, did not dismiss these concerns; instead, they sought to refine and reconsider the issues the mathematicians had raised, weighing against other professional obligations to determine their significance from their perspective. Consequently, teachers expressed alternative interpretations of the obligation to the discipline, often taking precedence over the mathematicians' view.

Our findings underline that the role of disciplinary values and principles in knowledge for SM teaching extends well beyond mere awareness. As Herbst and Chazan (2020) have observed, teachers need to interpret and weigh the significance of these values and principles in specific situations while considering other professional obligations. Thus, our study further underlines the necessity that mathematicians and teacher educators attend to teachers' disciplinary obligations not only in theoretical settings that are detached from the realities of SM education, such as academic mathematics courses, but also within the authentic context of classroom teaching.

Moreover, our study indicates that teachers should not be passive recipients but rather active participants in the articulation of the role and significance of the discipline in SM education. Furthermore, our findings suggest that there may be merit in mathematicians and teachers collaborating rather than working separately, as proposed for example, by Herbst and Chazan (2020). In our study, the mathematicians and teachers challenged each other's perspectives and pushed them to elaborate and refine them, and eventually negotiate not only whether the activities could be used but also how the activities could be used more responsibly from a disciplinary perspective while maintaining the pedagogical productivity the teachers recognized (cf. Pinto & Cooper, 2023). Such negotiations may pave the way for the development of a clearer and shared vision about the role and significance of the discipline in SM education.

Academic mathematics studies for teachers are intended to be a vital component of their professional preparation, equipping them with tools to teach school mathematics faithfully to the discipline. However, our findings indicate that presenting disciplinary obligations in a dichotomous way, without considering the authentic teaching context, may discourage teachers from fully integrating these obligations into their teaching. If our community aims to represent the discipline productively in teaching, the challenge lies in fostering a dialogue that accounts for both disciplinary and contextual obligations. Yet, this dialogue is often missing in current teachers' education frameworks. Further research is needed to investigate how such dialogue can be integrated into teacher preparation programs to better align disciplinary perspectives and professional obligations.

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