Mathematicians' Goals for Their Upper-Level, Proof-Based Mathematics Courses

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Despite significant research efforts focused on the development of student-centered instruction, most mathematicians continue to teach through lecture. Consequently, researchers have called for further examination of the reasons why mathematicians appear to believe in their current teaching practices. Our study builds on previous research addressing this call by examining how mathematicians' in- and out-of-class instructional decisions in upper-level, proof-based mathematics courses reflect their pedagogical goals. Four mathematicians participated in a series of three, one-to-one interviews examining their instructional design of a course they had recently taught. Data analysis is ongoing and is being guided by a deductive form of thematic analysis. In this paper, we provide an overview of literature on mathematicians' pedagogical goals, describe the study design and plans for data analysis, and present preliminary findings in relation to one participant mathematician.

Keywords: Upper-Level Courses, Goals, Resources, Orientations, Instructional Design

Introduction and Background Literature

In their recent review, Melhuish et al. (2022) distinguished between two forms of in-class instruction of undergraduate, proof-based, mathematics courses: those based on a lecture-based pedagogy and those based on a student-centered (or inquiry-based) pedagogy. Many mathematics education researchers have expressed dissatisfaction with lecture-based instruction, particularly its operation on a transmission-based model (Melhuish et al., 2022). Student-centered instruction, which emphasizes engaging students in doing mathematics during class time, has been suggested as an alternative (e.g., Laursen & Rasmussen, 2019). To assist instructors in adopting student-centered instruction, there has been significant research on the development of research-informed curricula for various mathematical content areas (e.g., Larsen et al., 2013; Leron & Dubinsky, 1995). However, "the influence of this research so far has been modest" (Melhuish et al., 2022, p. 4). Most mathematicians continue to teach through lecture (Johnson et al., 2018), despite the alternatives on offer. Additionally, the relatively small number of mathematicians who choose to adopt student-centered instruction are more likely to develop their own materials than to adopt research-informed curricula (Fukawa-Connelly et al., 2016).

The lack of uptake of these materials, despite the significant research efforts that focused on their development and dissemination, led to the call for research to "better explore the reasons that mathematicians appear to strongly believe in their current practice" (Fukawa-Connelly et al., 2016, p. 281). Subsequent research addressing this call (Woods & Weber, 2020) concluded that, in relation to lecture-based pedagogy, mathematicians choose to lecture as they believe in its efficacy as an instructional method and feel it aligns well with their pedagogical goals.

Over the past decade, numerous other studies have investigated various aspects of lecture-based instruction. This means we now have a good understanding of what happens in the 'typical' undergraduate mathematics classroom (Melhuish et al., 2022). However, the time instructors spend in class with students accounts for only one component of instruction. Many mathematics educators agree that, regardless of the format of in-class instruction, the majority of students' learning of advanced mathematics will take place outside the classroom (Rupnow et al., 2021). Therefore, an important aspect of teaching undergraduate mathematics is preparing students to work independently outside of class (Lew et al., 2022; Rupnow et al., 2021; Wu, 1999). This aspect of instruction has received much less attention in the mathematics education research literature. Rupnow et al. (2021) were among the first to examine how mathematicians assign homework problems in upper-level courses. Six mathematicians participated in task-based interviews in which they were asked to select homework problems from an abstract algebra textbook and justify their choices. The mathematicians' pedagogical goals guided their assignment of homework problems, and they perceived students' engagement with these problems as essential to their learning.

The studies of Woods and Weber (2020) and Rupnow et al. (2021) both address Fukawa-Connelly et al.'s (2016) call by providing insight into how mathematicians' decisions in relation to in-class instruction and homework are guided by their pedagogical goals. However, to date the various components of undergraduate mathematics instruction have mostly been examined independently, with a large emphasis on the in-class component. The goal of our study is to build on existing work and examine mathematicians' goals and pedagogical decision-making in relation to *all* instructional components of their upper-level, proof-based mathematics courses, along with the factors that influence these goals and decisions. Prior research that has surveyed mathematicians, has identified potential influences on their teaching practice, such as their experiences as a teacher and student (Johnson et al., 2018). However, research that has employed qualitative methodologies has demonstrated that mathematicians' teaching goals can vary on the level of the individual (e.g., Johnson et al., 2013; Pinto, 2019). Based on this research, we decided to adopt a qualitative study design to examine mathematicians' potentially varying goals and pedagogical decision-making in relation to all instructional components of their courses. In summary, our study addresses the following research questions:

- **RQ1:** What are mathematicians' pedagogical goals for their upper-level, proof-based mathematics courses and how do their orientations and resources inform these goals?
- **RQ2:** How are mathematicians' pedagogical goals reflected in the instructional decisions they make?

Theoretical Perspective

Given that research has indicated that mathematicians' instructional decision-making is primarily guided by their pedagogical goals (Rupnow et al., 2021; Woods & Weber, 2020), we decided to use Schoenfeld's (2010, 2011) theory of decision-making to inform our study design and data analysis. Schoenfeld proposes that the pedagogical decisions that a teacher makes can be understood as a function of their goals, resources, and orientations. Goals are described as "the things that people consciously or unconsciously set out to achieve" (Schoenfeld, 2011, p. 459) and can vary in terms of size. For instance, an instructor will likely have overarching goals for a course they are teaching, in addition to goals for an individual lesson. Schoenfeld suggests that a teacher's decision-making involves the selection of goals that are consistent with their orientations and resources. Orientations is used as an "inclusive term to encompass beliefs, dispositions, values, tastes and preferences" (Schoenfeld, 2010, p. 15). For example, over the course of their teaching career a mathematician may develop understandings or perceptions about the nature of mathematics, teaching, or students. Schoenfeld also proposes that the availability of resources also influences a teacher's decision-making. He elaborates that "what a teacher can or cannot do in the classroom is clearly a function of what he or she knows, what material and other resources are available, and what constraints are in place" (Schoenfeld, 2010,

p. 10). Schoenfeld's theory emphasizes the interaction among an individual's goals, orientations, and resources and notes that, typically, orientations and beliefs are slow to evolve and change.

Methodology

The study adopted a qualitative, phenomenological (Willig, 2021) research design. Four mathematicians participated in a series of three, one-to-one interviews in summer 2023. The interviews were designed to examine the mathematicians' instructional design of an upper-level, proof-based, mathematics course they had recently taught. Data analysis is in the initial stages.

Participants

Four research mathematicians, employed in a large, Irish, PhD-granting university, participated in the study. We employed purposive sampling (Braun & Clarke, 2013), inviting mathematicians with over 20 years' teaching experience who had taught an upper-level, proof-based mathematics course in the last three years to participate. We endeavored to have a balance in the research areas and institutions of undergraduate study of participants, as well as a gender representation that reflects that of the discipline of mathematics in the university. In this report, we present preliminary findings in relation to one participant to whom we assign the gender-neutral pseudonym Doolan.

Course Context

The four participants in the study were each interviewed about an upper-level, proof-based mathematics course they had recently taught. Given this paper's focus on findings related to Doolan, we provide further contextual details about their course. Doolan teaches a proof-based course covering topics from analysis to third- and fourth-year mathematics students. The in-class component of the course consists of 36 hour-long classes. Doolan lectures in roughly 30 classes, the remaining six classes are recitation classes (referred to as tutorials) in which the homework problems are discussed. There are four homework assignments in the course. Students are provided with a problem sheet consisting of around ten problems and are asked to submit solutions to two problems of their choosing. Each homework assignment is worth 5% of a student's overall grade, with a two-hour closed-book examination accounting for the remaining 80%.

Data Collection

All four mathematicians participated in a series of three, one-on-one interviews with the first author in summer 2023. Each interview lasted an average of an hour and was audio-recorded and transcribed. To add specificity to our interview questions, we asked each mathematician to select a section of content from their course consisting of at least one of each of the following: definition, example, theorem, and proof. We asked that the section of content they picked be reflective of the main focus of the course (rather than introductory material), and feature in the homework assignments.

The participants provided the lecture notes, homework assignments, and past examination questions relevant to their chosen section of content. These were used in the design of the interviews, with the first interview focused on the design of the lecture notes and lectures, the second on the homework assignments, and the third on the assessment component of the courses. As the focus of our preliminary findings is on homework, we provide further detail about the second interview. Based on the interview protocol used by Rupnow et al. (2021), for each problem related to the relevant content topic we asked why the mathematician had decided to include the problem and what they hoped students might learn from engaging with it. We also asked participants more general questions about how they operationalize homework in their courses and how they view the role of homework. Additionally, the mathematicians were asked if they hold recitation classes on homework problems and, if applicable, to describe the role and structure of these classes.

Data Analysis

Data analysis is in the initial stages. We plan to use a deductive form of thematic analysis, utilizing a method called template analysis (Brooks et al., 2015). Schoenfeld's (2010, 2011) theory of decision-making will inform the coding of the data. We also plan to incorporate the findings of Rupnow et al. (2021) and Woods and Weber (2020) in relation to mathematicians' goals as *a priori* codes (Brooks et al., 2015). To date, the authors have familiarized themselves with one participant's (Doolan's) interviews, with notes written on points of potential analytic interest. Analytic notes, and associated quotes, from the second interview were examined for evidence of pedagogical goals, as well as resources and orientations influencing these goals. This process informed the preliminary findings that we present in this paper.

Preliminary Findings

The aim of our study is to examine mathematicians' goals for their upper-level, proof-based mathematics courses, factors influencing these goals, and how their goals are reflected in their pedagogical decision-making in relation to all instructional components of their courses. In this paper, we present preliminary findings in relation to one of Doolan's pedagogical goals: encourage and support students in engaging with homework problems. We emphasize that data analysis is in the initial stages. Our primary purpose in presenting these findings is to illustrate the nature of our data and to present some of our initial analytic thoughts.

Doolan's reasons for assigning students homework problems broadly align with the findings of Rupnow et al. (2021). Like participants in Rupnow et al.'s study, Doolan downplays summative assessment as a purpose of assigning homework and highlights the learning opportunities engaging with problems can provide. Doolan emphasizes problem solving as a skill that should be developed in an undergraduate mathematics degree, stressing the importance of learning to struggle and persevere with non-trivial problems:

> *Doolan:* The whole point of it is that you hit these problems that you haven't met before, and you don't know if you can solve them. And if all you do is spend like a minute thinking about it and saying, you know, 'I can't do it' then, you know, you're not going to be a mathematician.

Given the importance they attribute to students' engagement with problems, it is perhaps unsurprising that one of Doolan's goals as an instructor is to encourage and support students in their work on homework problems. In this paper, we analyze one of Doolan's instructional decisions that we feel is consistent with this goal: the decision to allow students to select two homework problems from each of the four problem sheets in the course to submit for assessment. This decision appears to be primarily motivated by Doolan's pedagogical beliefs. Doolan proposes that by not assigning students particular problems, they will be encouraged to read, and engage with, all the problems to assess their relative difficulty:

> *Doolan:* If I give two particular problems as the assessment on that chapter, the students might just read those two problems and not bother about the rest. If they read the entire

sheet, even if it's with the viewpoint of finding the easy ones, they do have to think well, 'Is this hard or is this easy?' So, there's a little bit of learning going on just from that.

Additionally, by not requiring students to submit all problems for assessment, Doolan is able to include more difficult problems in the homework assignments. Acknowledging that it is likely that not all students will be able to complete these harder problems, Doolan believes there is still value in students attempting and thinking about them: "it would be a problem that I'd encourage students to think about, but I wouldn't expect them to hand it up for their assessment."

Doolan's decision to allow students agency in selecting their own problems also appears to have been influenced by their own experience as a student. Doolan recalled that as a student in a Ring Theory course, they became captivated by one question on a problem sheet: "one kind of intrigued me and I spent weeks like looking at it. I had a whiteboard at home and I was trying every kind of way." This formative experience as a mathematics student has left Doolan with the belief that "just throwing yourself into one problem like that, at the right time, can be really instructive. And, if it works out for you in the end, gives you great confidence." Doolan acknowledges that the decision to not assign students particular problems may be partly about "allowing a little bit of freedom" so that students have the chance to become similarly interested in problems of their own choosing.

We note that Doolan's decision to have students submit just two homework problems reflects the fact that they are constrained in terms of the time they have available to grade students' work: "practical things come into play. It takes quite a lot of time to grade. Yeah, so two is fair." We also highlight that Doolan's practice in relation to allowing students to select homework problems for assessment has evolved over the course of their teaching career. Doolan mentions that in previous iterations of the course they randomly assigned students particular problems to submit for assessment. The students at the time felt "it was a bit unfair that everyone had different problems to hand up." Based on this feedback from students, Doolan's understanding of students' perceptions of instructional practices developed, and their teaching practice in relation to assigning problems was adjusted accordingly.

Discussion and Conclusion

In this paper, we have outlined some of our initial analytic thoughts on one participant's goals and decision-making in relation to homework. While emphasizing the preliminary nature of our results, we note the apparent alignment between Doolan's instructional decisions and their pedagogical beliefs. This is consistent with prior research indicating that mathematicians can generally offer sophisticated rationales for their pedagogical actions (Melhuish et al., 2022). The implication of this finding is that attempts to change mathematicians' practice are likely to fail if the alignment between instructional interventions and mathematicians' goals is not considered (Melhuish et al., 2022; Woods & Weber, 2020). Consequently, developing a robust understanding of mathematicians' pedagogical goals and decision-making in relation to both in- and out-of-class instruction is important in terms of informing the development of instructional interventions. In this regard, our ongoing work on this study builds on previous research by examining mathematicians' goals and pedagogical decision-making in relation to all instructional components of their upper-level, proof-based courses.

Acknowledgements

The authors would like to acknowledge financial support from the University College Dublin College of Science Seed-Funding Scheme which facilitated their participation at the conference.

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