Student Understanding of Solids of Revolution: Refining a Previous Model

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In this study, we consider students' performance on tasks involving solids of revolution to refine a recently introduced, research-based model. An activity was designed with tasks to help students do the constructions proposed in the model. After implementing the activity in the classroom, we suggest additions to the model using data from the written responses of 22 university calculus students' quiz and exam tasks. The results provide information about the challenges that students face when dealing with solids of revolution and allow us to refine the model to add more detail that might be used to help students overcome observed challenges. We discuss some limitations of the study and share teaching implications.

Keywords: Solids of revolution, Calculus, APOS theory, Riemann sums

Introduction and Literature Review

There is considerable research on student understanding of one-variable function integration (e.g., Jones, 2015; Orton, 1983; Sealey, 2014; Thompson, 1994; Thompson & Silverman, 2008). In one of the first studies dedicated to integration, Orton (1983) found that volumes of revolution were one of the three most difficult integration topics for students; the others were Riemann sums and limits of Riemann sums. Jones (2013) observed three major meanings for symbolic forms of integrals that emerged from student data: adding up pieces, perimeter and area, and function matching that roughly correspond respectively to a definite integral's relation to Riemann sums, the area-under-a-curve conception, and the integral as an anti-derivative. Jones (2013) observed the importance of the "adding up pieces" interpretation for applications and, like Orton (1983), suggested it needed more emphasis. He later reported (Jones, 2015) that the area-under-a-curve and anti-derivative conceptions were highly prevalent, with the Riemann sum-based conception being underutilized. This underutilization was also observed by Akrouti (2020), even in tasks where using Riemann sums was the only reasonable approach. Speer and Kung (2016) observed that there is little research dealing with volumes of solids of revolution (SoR). Some studies (e.g., Paul et al., 2024) consider the use of manipulatives when teaching this topic. However, few studies unpack students' difficulties with the topic. In one such article, Mofolo-Mbokane et al. (2013) found that students had struggled with graphing skills, going from continuous to discrete (drawing approximating rectangles and visualizing rotation to decide whether to use dx or dy), translating from 2D to 3D diagrams (producing drawings), manipulation skills (computing the definite integral), and general cognitive skills (putting everything together). In another study dedicated to SoR, Andrade and Montecinos (2013) argued that students need help to interpret $f(x)$ as something more than just "the function to be integrated," coinciding with Sealey's (2014) need for an orienting pre-layer. They suggested, as did Blanco et al. (2019), that students need support to generate an image of a 3D solid starting from a 2D drawing. Given students' difficulties in visualizing SoR, Juárez and colleagues (2020) designed a questionnaire to evaluate students' skills in visualizing such solids. Later, Juárez and different colleagues (2022) designed a GeoGebra intervention to improve students' visuospatial abilities when dealing with SoR. Bernard and Jones (2016) studied the steps students chose when computing several volumes of SoR, the order of the steps, and the reasoning for this order. Their findings suggest that visualizing the rotated solid may be a factor that influences students' ability to solve SoR problems that should be explicitly addressed. They also found that some students might not relate the differentials in the integral expression with the thickness of disks or washers and that students frequently rely on memorized procedures in setting up integral expressions. These observations underscore the need to help students visualize SoR and relate Riemann sums to definite integrals, as is done in the present study.

The research goal for this study was to use data collected from students' performance on SoR tasks to refine and suggest revisions for the theoretical model proposed by Borji and Martínez-Planell (2023) to explain how students come to understand volumes of revolution and what obstacles they might face in this process. We designed and implemented an activity to help students make the constructions proposed in the model. The results are discussed, and a revision of the model is proposed.

Theoretical Framework

We use Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014), focusing only on the A, P, and O conceptions. An *action* is a transformation of a previously constructed mathematical notion that a student perceives as external, in the sense that, due to the lack of connections to other mathematical knowledge, the transformation is carried out only by memorization or following explicit instructions. When an action is repeated and reflected upon, it might be interiorized into a process. The *process* is perceived as internal since connections to other knowledge have been established, allowing students to anticipate results, generate dynamical imagery, make justifications, and perceive it as independent of representation. Different processes can be coordinated to form new processes. When a student can think of a process as an entity in itself and perform actions on it then one says that the process has been encapsulated into an *object*.

An important idea in APOS is that of a genetic decomposition (GD). This is a model of how a student may construct a specific mathematical notion. A GD is expressed in terms of the theory's structures (actions, processes, objects) and mechanisms (interiorization, encapsulation, coordination). A GD of a given notion starts as an initial hypothesis that must be tested with student data. Experimentation can lead to its verification, refinement, or rebuttal. An initial GD for SoR proposed by Borji and Martínez-Planell (2023) considered only the disk and washer methods. We will briefly summarize it here, and then, at the end of this paper, we propose a revised version that also applies to shells. The GD suggested by Borji and Martínez-Planell, which we take as our initial GD, includes prerequisites (i.e., understanding of function, area of a circular region, geometry and volume of a right circular cylinder) and has three parts:

- GD1: Geometric SoR constructed by rotating points, curves, and regions around a given axis; this allows imagining and visualizing SoR
- GD2: Geometric approximation constructed by forming small partitions, rotating the rectangles in the partition, drawing the resulting disks or washers, and comparing with the SoR being approximated; it allows imagining the approximation of the volume with an increasingly large, but finite, number of disks or washers
- GD3: Geometric-analytic volume of the SoR constructed by small partitions expressing the corresponding volume numerically, then symbolically as an extended sum, using sigma notation, then using an infinite number of disk or washers to determine the volume by taking limits and finally expressing the limit as an integral; it allows relating different representations for volume including the limit of Riemann sums and integrals

Methods

The three authors collaboratively designed an activity (https://bit.ly/4a9UNOX) with tasks 1-4 involving rotation only (GD1); tasks 5-8 involving rotation and rotation corresponding to a single subinterval of a partition for volume approximation (GD1, GD2); and tasks 9-11 involving rotation of a curve, volume approximation with limits, and volume with integration (GD1, GD2, GD3). The activity was administered in a multivariable classroom in a Southwestern US university for bonus points with 22 students who were allowed to consult with others on the assignment and to take it home to complete, if needed. The first two authors individually combed through the activity data and scored each task part using 0 (completely incorrect), 1 (somewhat correct), 2 (mostly correct), and DNA (did not attempt). The scoring agreement was 96%; the research team discussed scoring until a consensus was reached.

The data from the activity tasks were used as a first pass to study the students' understanding of SoR. Next, the first two authors jointly revisited the description of the GD of SoR found in the Borji and Martínez-Planell (2023) paper and then individually reviewed the quiz and exam task data making notes of their observations.

We also considered quiz and exam tasks (https://bit.ly/4a9l0wZ) that dealt with SoR that required the use of washers and disks. The coding scheme for the quiz and exam tasks with descriptions of observed student struggles appears in Table 1. The codes were developed jointly by the first and second authors as they combed through the data. The correctness scoring for all tasks in the activity and GD coding for the quiz and exam student data were used to complete qualitative profiles in terms of GD1, GD2, and GD3 for each student's understanding of SoR. The third author observed the students in class, reviewed all data individually, and provided input on the reported results.

Student Performance

Comparison of Performance on Activity Tasks with Quiz and Exam Tasks

First, we compare student performance scores on the activity tasks divided into GD1 (imagining SoR), GD2 (imagining rotated rectangles approximating volumes of SoR), and GD3 (converting from geometric imagery to integral expressions). Students tended to perform better with higher average scores on tasks that required GD1 only (mean: 92%, median: 100%) than on tasks requiring GD1 and GD2 (mean: 81%, median: 92%). Similarly, they performed better on tasks requiring GD1 and GD2 than those requiring GD1, GD2, and GD3 (mean: 45%, median: 41%). Activity tasks 10 and 11, which required GD1, GD2, and GD3, had 8 parts each. Of the 352 possible responses for all 22 students on the 16 parts of these two tasks, 46% were left blank (coded DNA) and another 19% were scored 0.

GD Coding

Since the activity allowed for collaboration on its tasks, the quiz and particularly the exam task data (since the exam was administered at the end of the semester) were primarily used to create a profile for each student in terms of the GD of SoR. Each researcher independently recorded this in individual notes for the student data. Upon inspection of the student responses to the tasks, the research team noted that students often struggled with prerequisite skills that had not been identified earlier. The data prompted discussion to conclude that there were additions needed to the original GD of SoR. Table 1 below summarizes the most prevalent student struggles observed on quiz and exam tasks by GD 1, GD2 and GD3.

Table 1. Codes Used for Prevalence of Student Struggles on Quiz and Exam Tasks by GD1, GD2, GD3

	Codes for Student Struggle (prerequisites denoted)	Description of Student Struggle	Frequency of Struggle
GD1	*Rotation*	Difficulty rotating or missing rotation of curve or region	20
	Axis (Prereq)	Rotation around incorrect line or axis	2
	Region (Prereq)	Difficulty determining region to be rotated when given equations that defined region	2
	Graph (Prereq)	Difficulty graphing a curve given as an equation	1
GD2	*Partitions*	Difficulty creating or failure to create rectangles to partition region	20
	Orientation	Incorrect orientation for partitions	3
GD3	*Cross-sect Area (Prereq)*	Incorrect cross-sectional area of disk or washer	24
	Radius (Prereq)	Incorrect radius for disk or washer	21
	Differential	Difficulty with differential or lack of correspondence between differential and partition	17
	Limits of Int	Difficulty with limits of integration	9

The different coded observations suggested needed refinements for the GD. For GD1, the *rotation* code is not useful since a student might not produce an image on a task and yet be able to imagine the rotated solid. While this would be less of an issue in an interview, it was a limitation of having data collected via paper-and-pencil instruments. So, not drawing a rotation might mean that the student could visualize the drawing mentally; however, it might also indicate a reticence of the student to use a geometric argument. The *axis* code denoted observations suggesting lack of a prerequisite construction to identify the axis of rotation. This prerequisite was not mentioned in the initial GD by Borji and Martínez-Planell (2023). The codes *graph* and *region* seem to result from a lack of prerequisite knowledge. Students did not correctly draw the graph of one-variable functions or determine regions bounded by one-variable functions and lines. As with the *axis* code, this was not mentioned as a prerequisite in Borji and Martínez-Planell's GD. For GD2, the *partitions* code may tell us that a student did not draw correct rectangles to partition the region but might be able to imagine the approximating rotated rectangles. It may also signal students' reticence to use geometric methods. There were also observations with *orientation* codes that all occurred when the students tried to use shells rather than washers for the problem. The initial GD did not consider shells. The revised GD describes the constructions needed for the shell method, although we are not considering such data in this paper. The many observations with the *radius* and *cross-sect area* codes moved us to add prerequisite constructions for radius and area as well as to also change GD2 to include new constructions that will have students rotating "thin" rectangles of infinitesimal width *dx* or *dy* around horizontal or vertical axes, as well as computing volumes of the disks, washers, or shells that are formed. These changes in GD2 can also help students avoid issues described in the *differential* code, and together with changes in GD3, they can also avoid those in the *limits of int* code. Now, we consider an example to show how the coding was assigned.

Figure 1 shows Student 14's work to determine the volume of the SoR obtained by rotating the region bounded by $y = -x^2 + 9$, $y = 4$, $x = -1$, and $x = 2$ about the x axis. We coded his work GD1, since he drew the boundary curves and lines as well as a rotation symbol to the right of the x axis suggesting he imagined the SoR. His work was coded GD2 since his sketched rectangle suggests he imagined rotated rectangles approximating the volume of the SoR. His work was coded no GD3 since there were errors converting from geometric imagery to an integral expression. He used $(-x^2 + 5)$ rather than 4 for the inner radius; so, it was coded *radius*. Consequently, the cross-sectional area is also wrong and thus coded *cross-sect area*. It was also coded *limits of int* since the limits of integration were 4 to 9 rather than -1 to 2.

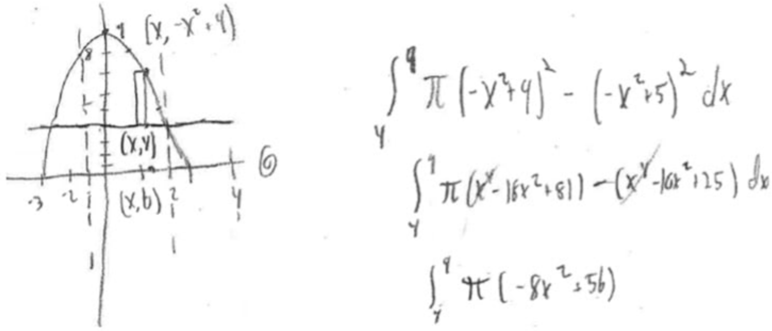

Figure 1. Student 14's response to one of the quiz tasks.

See Figure 2 for Student 15's work on the same problem. His work was coded as no GD1, and given the *region* code, for choosing the wrong region to rotate. It was coded as no GD2 and given the *orientation* code for drawing the approximating rectangle with the wrong orientation. It was coded no GD3 and assigned both the *radius* and *cross-sect area* codes since the written integral misses the radii and cross-sectional area of a washer and instead refers to a disk that is unrelated to the drawing.


Figure 2. Student 15's response to one of the quiz tasks.

Implications for Teaching Practice

Codes no GD1 and no GD2 might point to some student reticence to use geometric methods to find volumes of revolution. This may be a result of inconsistent participation in the GD-based homework tasks. Many students did not attempt the totality of the assigned tasks and may not have taken the tasks that required drawings seriously, instead producing very small, incomplete, seemingly careless sketches. This suggests that the students might have benefitted from the incorporation of the ACE (**A**ctivity in collaborative groups-**C**lassroom discussion-**E**xercises at home) pedagogical strategy that is an integral part of APOS theory. This pedagogical strategy foments student reflection, which is needed to interiorize actions into processes and encapsulate processes into objects. Awarding more credit than just a few bonus points to students for their work might have a positive effect on their motivation to complete GD-based tasks. So, the way in which the GD-based tasks are implemented is likely to play an important role.

The results also allow us to revise the genetic decomposition to include more detail. As observed in the results, we need to revisit the prerequisite knowledge of students to construct processes to identify axes, graph one-variable functions (including vertical and horizontal lines), and identify regions in the plane that are described in terms of graphs of functions and lines. The description of GD1 needs to include constructing a process to communicate (i.e., produce words or drawings) to describe a rotated planar region. Regarding GD2, a prerequisite process to recognize cylinders, washers, and shells (obtained from rotating "thin" rectangles) is needed, as is distinguishing between the area and volume of common geometrical shapes. GD2 must be revised to make explicit the construction of processes to visualize approximating the area of a planar region with rectangles, all of which go from a given curve to another and account for the entire region, visualize the rotated rectangles as disks, washers, or shells; and recognize that the rotated rectangles are an approximation of the corresponding rotated region. Regarding GD3, even though most students did not attempt many of the corresponding tasks, students' work on the quizzes and exams suggests that the GD needs to be explicit about constructing prerequisite processes for the radius, circumference, area of a circle, as well as the volume of disks, washers, and shells. The description of GD3 also needs to be explicit about constructing processes to determine the radius of disks, washers, and shells; determining the height of shells; thinking of *dx* or *dy* as an infinitesimal thickness related to a partition of an interval, computing cross-sectional areas, relating the partitioned interval to the limits of integration, and in general, bridging the geometric and analytical representations from a finite to a generic Riemann sum and from this to imagining and expressing the limit of Riemann sums as an integral.

Revised Genetic Decomposition

The study of students' work suggests refinements are needed for Borji and Martínez-Planell's (2023) GD. The resulting revised genetic decomposition (see Table 2) addresses disks and washers as well as extending to also include shells.

Table 2. Revised Generic Decomposition for Solids of Revolution

GD0: Prerequisite ideas	• Understanding[1] of one-variable functions including algebraic and graphical representations • Understanding of horizontal and vertical lines including axes • Ability to identify planar regions defined by functions and lines whether given graphically or algebraically • Understanding of the difference between length, area and volume • Understanding of circles, circular regions, and cylinders, including related features such as radius, circumference, and surface area (including lateral area) • Understanding of volumes of disks, subtraction of volumes of figures when one is "inside" another (such as is the case with washers), volumes of shells • Understanding of sigma notation as a representation of summation • Introductory understanding of a limit of a length being zero
GD1: Geometric solid of revolution	• The action of rotating a single point around an axis is interiorized into a process of point rotation that allows imagining the resulting circle without having to do the rotation explicitly. The need to rotate curves and regions (viewed as collections of points) around an axis foments the encapsulation of the point rotation process into a point rotation object. • The action of rotating a curve around an axis and drawing the resulting surface can be interiorized into a process of curve rotation. The need to draw or imagine surfaces of revolution foments the encapsulation of the process of curve rotation into an object of curve rotation. • The action of rotating regions bounded by one or more curves of the form y=f(x) or x=f(y), one complete cycle around an axis to form a solid, drawing the resulting solid, and describing it verbally, is interiorized into a process that allows the student to imagine the formation of such solids of revolution and describe it verbally and with a drawing. The need to do actions on solids of revolution, such as computing their volume, foments their encapsulation into objects.

[1] By "understanding" a notion, we mean having constructed at least a process of the notion.

GD2: Geometric approximation by disks, washers, or shells	• The action of rotating a "thin" vertical (horizontal) rectangle of infinitesimal width *dx* (or *dy*) and with length that is either a number or a symbolic expression, rotating it around a given axis (any vertical or horizontal line not intersecting the interior of the rectangle), drawing the resulting "thin" solid and describing it verbally, can be interiorized into a process that allows imagining the formation of infinitesimally thin disks, washers, and shells. • This process can be coordinated with prerequisite processes in order to compute the resulting volumes (numerically and symbolically). • Coordination of these processes results in a process of disk, washer, shell formation that interrelates geometric, algebraic, numerical, and verbal representations. • Given a planar region defined by one or more curves of the form y=f(x) or x=f(y) and given (or having the student produce) the corresponding solid of revolution, the action of determining an interval and forming a partition of the interval with a small number (an explicit number, not a generic symbol) of subintervals and drawing the corresponding rectangles, disks, washers, or shells to approximate the volume of the solid of revolution is interiorized into a process that allows a student to imagine approximating a region with a small number of rectangles, imagine the rotated rectangles as disks, washers, or shells, and compare the volume of the rotated rectangles to that of the solid of revolution. • This process can be coordinated with the process of disk, washer, and shell formation to form a new process that allows the student to imagine partitions of an interval and corresponding planar region into many thin rectangles and approximate the volume of a solid of revolution with an increasingly large corresponding number of solid rings, washers, or shells without having to do so explicitly.
GD3: Geometric-analytic volumes of revolution	• The chain of actions of first writing the numerical approximation given by the Riemann sum corresponding to the solid obtained with a small partition in GD2 (for example, taking $\Delta x = 1$, for the solid obtained by rotating $y = x^2$ from $x = 0$ to $x = 3$ it could be something like $\pi(1^2)(1) + \pi(4^2)(1) + \pi(9^2)(1)$), then introducing variables and labeling a drawing of the previously obtained solid and writing the Riemann sum in extended form (for example, something like $\pi({x_1}^4)\Delta x + \pi({x_2}^4)\Delta x + \pi({x_3}^4)\Delta x$), then writing the Riemann sum using sigma notation (for example, something like $\sum_{i=1}^{3} \pi x_i^4 \Delta x$), before generalizing and finally taking limits ($\lim_{n\to\infty} \sum_{i=1}^{n} \pi x_i^4 \Delta x$) and representing and computing the limit as an integral ($\int_0^3 \pi x^4 \, dx$) may be interiorized into a process. • This process allows imagining the sum of volumes of a finite number of disks, washers, or shells as a numeric approximation to the volume of revolution (bridging the numeric with the geometric); allows imagining a partition of the region into rectangles of width Δx and use variables x_i rather than numbers (bridging the numeric and algebraic) to obtain and sum the corresponding volumes. • When this process is coordinated with the process resulting from GD2, the resulting process allows to imagine that a general partition into an increasing number *n* of thinner and thinner rectangles results in a better approximation (bridging the algebraic with the geometric), allows to express general Riemann sums in sigma notation and recognize the limit as the value of the volume of revolution, and allows to express the limit of Riemann sums as an integral. • The need to take the limit of such finite sums foments the encapsulation of the process into an object that identifies a limit of Riemann sums for the volume of a solid of revolution with the corresponding volume integral.

The GD-based tasks used should be revised to reflect the new genetic decomposition. They need to be class-tested using the ACE pedagogical strategy, and a new research cycle needs to be undertaken. Interviews are needed to help capture students' thinking about SoR, their meanings for the different notions involved, and the reasons for their problem-solving actions.

References

Akrouti, A. (2020). Conceptions de l'intégrale de Riemann des étudiants à l'entrée en classes préparatoires [Riemann integral conceptions of students entering preparatory classes]. In T. Hausberger, M. Bosch, and F. Chellougui (Eds.). *Proceedings of the Third Conference of the International Network for Didactic Research in University Mathematics*, p. 53–62. Bizerte, Tunisia: University of Carthage and INDRUM. https://indrum2020.sciencesconf.org/data/pages/INDRUM2020_Proceedings.pdf

Andrade Molina, M., & Montecino, A. (2013). Conversión de registros en el cálculo integral: la problemática de los sólidos de revolución [Conversion of registers in the integral calculus: the problem with solids of revolution]. In R. Flores (Ed.), *Acta Latinoamericana de Matemática Educativa*, *26* (Capítulo 1), pp. 473–479). México, DF: Comité Latinoamericano de Matemática Educativa. https://core.ac.uk/reader/20482520

Arnon, I., Cotrill, J., Dubinsky, E., Octaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. Springer Verlag. https://doi.org/10.1007/978-1-4614-7966-6

Bernard, A., & Jones, S. R. (2016). Student problem solving in the context of volumes of revolution. In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*, Pittsburgh, Pennsylvania, (pp. 554–561). Retrieved from http://sigmaa.maa.org/rum/RUME19v3.pdf

Blanco, T. F., Diego-Mantecón, J. M., & González Sequeiros, P. (2019). Procesos de visualización en una tarea de generación y representación de cuerpos de revolución [Visualization Processes in a Task that Requires Generating and Representing Solids of Revolution]. *Bolema: Boletim de Educação*, *33*(64), 768–789. https://doi.org/10.1590/1980-4415v33n64a16

Borji, V., & Martínez-Planell, R. (2023). On students' understanding of volumes of solids of revolution: An APOS analysis. *Journal of Mathematical Behavior, 70*, 101027. https://doi.org/10.1016/j.jmathb.2022.101027

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*, 122–141. https://doi.org/10.1016/j.jmathb.2012.12.004

Jones, S. R. (2015). The prevalence of area-under-a-curve and antiderivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals, *International Journal of Mathematical Education in Science and Technology, 46*(5), 721–736. https://doi.org/10.1080/0020739X.2014.1001454

Juárez, E. L., Hernández, C. A., Slisko, J., & Garcia, R. (2020). Instrumento diagnóstico para la evaluación de habilidades espaciales en sólidos de revolución: Validación de contenido. [Diagnostic instrument to evaluate spatial skills for solids of revolution: Content validation]. In L. A. Hernández Rebollar, G. Kantún Montiel, y J. Slisko Ignjatov (Eds.), *Tendencias en la Educación Matemática 2020* (Capítulo 1, pp. 97–115). Benemérita Universidad Autónoma de Puebla. https://www.researchgate.net/publication/347506068

Juárez-Ruíz, E., Sánchez González, L., & Juárez López, J. A. (2022). Identificación del desarrollo de habilidades visuales espaciales en representaciones y conversión entre registros para calcular volúmenes [Identification of the development of visuospatial abilities in the representation and conversion between registers to calculate volumes]. *Educación Matemática*, *34*(1), 157–185. https://doi.org/10.24844/EM3401.06

Mofolo-Mbokane, B., Engelbrecht, J., & Harding, A. (2013). Learning difficulties with solids of revolution: classroom observations. *International Journal of Mathematical Education in Science and Technology, 44*(7), 1065–1080, https://doi.org/10.1080/0020739X.2013.823253

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics, 14*, 1–18. https://doi.org/10.1007/BF00704699

Paul, S., Grundmeier, D., & Moore-Russo, D. (2024). 3D manipulatives in integral calculus: Student achievement and confidence in solids of revolution tasks. *Investigations in Mathematical Learning*, *16*(2), 100–119. https://doi.org/10.1080/19477503.2024.2312097

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*, 230–245. https://doi.org/10.1016/j.jmathb.2013.12.002

Speer, N., & Kung, D. (2016). The complement of RUME: What's missing from our research? In T. Fukawa-Connelly, N. Infante, M. Wawro, and S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1288–1295). Pittsburgh, PA. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html#:~:text=Full%20Proceedings%20(PDF%2C%20117%20MB)

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*, 229–274. https://doi.org/10.1007/BF01273664

Thompson, P., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 117–131). Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.005