Examining Subjective Factors Shaping Students' Combinatorial Thinking

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Research on combinatorics education indicates a crucial need in mathematics education to address students' ways of thinking at a level that benefits a deeper understanding of how students conceptualize counting problems. Attending to the ways in which students approach and interpret counting problems reveals the subjectivity in their decision makings. In this study, I report data from written works submitted by thirty five prospective high school teachers as they considered a disagreement between two possible responses to a combinatorial problem. Defining subjective combinatorics as a subjective aspect of students' interpretations of sets of outcomes facilitates describing students' counting activity. I propose subjective combinatorics as a construct to shed light on our understanding of students' counting activities, focusing on the subjectivity inherent in their combinatorial thinking.

Keywords: combinatorics, conceptual analysis, counting problems, undergraduate mathematics education, subjective combinatorics

Enumerative combinatorics, the area of mathematics focused on solving counting problems, is part of many K-12 and undergraduate mathematics curricula. This prominence is justified, as counting problems not only have practical applications in fields like computer science and probability (e.g., Abrahamson, Janusz, & Wilensky, 2006; Shaughnessy, 1977) but also challenge students to engage in critical mathematical thinking (e.g., Kapur, 1970; Martin, 2001; Tucker, 2002). In recent years, the mathematics education research community has placed greater focus on students' combinatorial reasoning. Numerous studies have examined common errors, strategies, and different ways of thinking involved in solving counting problems (e.g., Annin & Lai, 2010; Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004; English, 1991, 2005; Halani, 2012; Lockwood, 2011, 2013; Lockwood, Swinyard, & Caughman, 2015; Tillema, 2013). Despite the critical role of combinatorial reasoning, many of these studies highlight the difficulties students face in solving counting problems. Given these difficulties, it became apparent to investigate the factors that influence students' success in combinatorics. Understanding these factors requires a detailed examination of the elements involved in students' combinatorial thinking. To this end, research on combinatorics education has progressively focused on understanding students' ways of thinking about counting problems. Lockwood (2013) conceptualized students' combinatorial thinking by identifying three primary components: formulas/expressions, counting processes, and sets of outcomes. This model has significantly enhanced educators' ability to analyze and understand students' combinatorial thinking. This study aims to focus on how personal interpretations and subjective judgments impact students' combinatorial thinking. By investigating this dimension, I propose subjective combinatorics as a construct to better understand the role of subjectivity in students' combinatorial thinking.

Literature Review and Contextual Insights

This research focuses on subjectivity in students' combinatorial thinking. I am not aware of any research that attends to this explicit focus. As such, in this section I turn to prior research on the related topics; I first describe several studies on subjectivity and then focus on research on students' combinatorial thinking.

On Subjectivity

To explore how subjectivity can influence the process of solving counting problems, it is essential to start by considering the definition of counting problems themselves. A counting problem highlights restrictions in representing a set of outcomes and tasks students with determining the cardinality of the set of outcomes. Expanding upon the definition of counting problems, the highlighted restrictions can be presented either explicitly or implicitly. When restrictions are given implicitly, they can be interpreted subjectively. For example, in counting the number of outfits when there are two shirts, five pairs of pants, and three hats, the word "outfit" could be interpreted subjectively. For one student, having a hat as part of an outfit might be a necessity, while another student might find a shirt and pants sufficient. As exemplified, the word "outfit" has many distinct interpretations. Hence, students' answers will depend on how they interpret "outfit". Since a counting problem can be interpreted differently from one person to another, the set of outcomes that needs to be counted could therefore be different.

Attending to the ways in which students specify outcomes while solving counting problems highlighted the subjectivity in their combinatorial thinking. This led me to consider a new aspect of combinatorics, to which I refer as *subjective combinatorics*. Although the subjective aspect of some areas in mathematics such as probability has been considered in the mathematics education literature (Chernoff & Zazkis, 2011; Jones, Langrall, & Mooney, 2007) the subjective aspect of combinatorics has not been studied explicitly.

According to Jones et al. (2007) "*subjective probability,* describes probability as a degree of belief, based upon personal judgment and information about an outcome" (p. 913). Batanero et al. (2005) highlighted "in this subjective view, what is random for one person might be nonrandom for another…a degree of uncertainty specific for each person" (p. 24). Borovcnik et al. (1991) clarified "the basic assumption here is that subjects have their own probabilities which are derived from an implicit preference pattern between decisions" (p. 41). According to Hawkins, & Kapadia (1984) "to a greater or lesser extent the probability is an expression of personal belief or perception" (p. 349). Combinatorics, like probability, involve a degree of subjectivity influenced by the problem itself and the solver's knowledge and experience. I introduce the notion of *subjective combinatorics* and in what follows, I provide further motivation for this construct. I make a case that subjective combinatorics provides a valuable lens through which to study the learning and teaching of combinatorics. In particular, I argue that recognizing subjectivity in students' combinatorial thinking is beneficial to enhance our understanding of how students conceptualize counting problems. To this end, my research aims to develop a detailed description of subjective combinatorics and to explore its potential effect on students' combinatorial thinking. This dual focus aims to provide evidence for – and to lay groundwork for – future studies to examine subjectivity in students' combinatorial thinking. In particular, this study attempts to answer the following research questions: *How does subjectivity affect students' combinatorial thinking? How do students generate restrictions as they solve and evaluate counting problems that are susceptible to be considered subjectively?*

On Students' Combinatorial Thinking

Lockwood (2013) introduced a model of students' combinatorial thinking that included three components: formulas/numerical expressions, counting processes and sets of outcomes. Building on Lockwood's model, this study explores how subjectivity influences students' combinatorial thinking, particularly in their interpretation and generation of sets of outcomes. The concept of subjective combinatorics, introduced in the following section, serves as a guiding theoretical principle, with a central focus on the role of personal judgment in interpreting.

Theoretical Framework: Subjective Combinatorics

Drawing from the concept of subjective probability, *subjective combinatorics* views combinatorics as a matter of belief, shaped by personal judgment and information about outcomes. It depends on two things: the combinatorics problem and the knowledge and experience of the person solving the problem. Just as in subjective probability, where the likelihood of an event is assessed based on personal perspectives, subjective combinatorics acknowledges that different solvers may approach counting problems with varying assumptions, leading to different yet potentially valid solutions. This construct emphasizes the variability in how individuals conceptualize and reason through combinatorial tasks, which have been shaped by their experiences, understanding of the problem, and the context in which the problem is presented. I propose that the differences between "correct" and "incorrect" responses to counting problems, which stem from different subjective interpretations, are merely differences in the description of sets of outcomes. To determine the specific interpretation applied among the possible ones, insights can be gained by examining the comments from individuals who have solved the problem.

In daily life context problems, students might need to navigate two different interpretations simultaneously: one based on their understanding of the problem's requirements and another shaped by their personal experiences and prior knowledge. Balancing these interpretations can be challenging as they may need to reconcile what the problem explicitly asks with how they relate to it from their own experience. The notion of subjective combinatorics can shed light on understanding these diverse interpretations.

Methods

In this study, as part of a larger study, thirty-five preservice secondary mathematics teachers participated. At the time of data collection, they were enrolled in a mathematical methods course for teaching secondary mathematics within a professional development program in mathematics education. While their mathematics background and experience varied significantly, all held degrees in mathematics or science. The course is typically completed in the final semester of studies toward teaching certification. In this course, students are expected to enhance their personal mathematical knowledge and investigation skills, and to explore pedagogical approaches to secondary mathematics. The focus is on topics often overlooked in school but not requiring knowledge beyond the school-level mathematics.

Counting problems was one of the chosen topics for exploration. During class sessions, the participants were either reminded of or refreshed their knowledge of the multiplication principle and addition principle from their high school studies and discussed combination, permutation, the binomial theorem, and combinations with and without repetition. The students revisited different counting problems and discussed possible errors in counting problems. After these discussions, participants were given the following task:

You work with a class on counting problems. One of the problems asks students to solve the *Kittens problem*: There are 10 kittens up for adoption. How many different ways are there to distribute kittens to 7 people so that everyone gets at least one kitten and at most two kittens? Two students suggest two different answers.

Jamie's answer: $\binom{7}{3}$

Lucas's answer: $\binom{7}{3} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times 4!$

1) Which answer do you agree with? Explain your reasoning. If you disagree with both, please provide your own answer and justify your reasoning. 2) What do you think was implicitly

assumed in each answer? 3) Why do you think the students made these assumptions? 4) Reflect on your engagement with the assignment: How have you dealt with it? Did you have any difficulties and/or dilemmas while working on the assignment? What have you learned through the process? The suggested length for responding to each question is about 150 words.

The *Kittens problem* was intentionally designed to allow for subjective interpretation, making it a suitable task for exploring subjective combinatorics. Some students might view the kittens as identical, while others might treat them as distinct entities. Additionally, one interpretation might require that all 10 kittens be distributed among the 7 people, whereas another might allow for scenarios in which only 9, 8, or 7 kittens are adopted. In the latter case, this would result in 2, 1, or 0 people receiving an extra kitten, respectively. Both suggested responses in the task are correct based on different interpretations. Jamie's and Lucas's responses suggest consideration of distributing 10 kittens among the 7 people, however, they do not share the same consideration for kittens. Jamie's answer assumes that the kittens are identical and focuses on choosing 3 out of the 7 people who will each receive 2 kittens. Since Jamie considers the kittens to be identical, there is no need for any additional counting processes or any additional consideration of the set of outcomes. Lucas's approach builds on Jamie's by considering the kittens as distinct entities. While Lucas also starts by choosing 3 out of 7 people to receive 2 kittens each, he incorporates additional steps to account for the distinct nature of each kitten. Lucas counts the number of ways to assign 2 distinct kittens to each of the 3 chosen people and then distributes the remaining 4 distinct kittens to the remaining 4 people. This involves considering $\binom{10}{2}$ for the first person, $\binom{8}{2}$ for the second, $\binom{6}{2}$ for the third, and 4! for the final distribution.

The participants' written works comprise the data for this study. The claims highlighting subjective interpretations served as a unit of analysis. Having repeatedly read the data, I looked for common themes by highlighting noteworthy explanations. I then categorized them based on participants' views on the uniqueness or interchangeability of kittens. This process involved examining how unconscious assumptions, real-world experiences, personal and mathematical connections, and level of detail influenced participants' interpretations.

Results

In the following analysis, participants' interpretations of the problem, whether viewing the kittens as identical or unique, are presented along with their reasoning. While the excerpts primarily focus on participants' interpretations and their reasoning, it is important to note that these interpretations directly informed their combinatorial thinking. Specifically, participants' interpretations shaped their understanding of sets of outcomes and their counting processes, as reflected in their alignment with either Jamie's or Lucas's approach.

Interpretation 1: Kittens as Identical

Unconscious assumptions. In solving counting problems, personal assumptions often play a significant role in shaping how individuals interpret and solve problems. The following excerpt illustrates this well.

> *Shawn:* I could immediately understand Jamie's answer because it matched my initial thinking. I found Lucas's answer very perplexing until I discussed it with my friend. He mentioned that the kittens could be considered unique or interchangeable. That is when I realized that I had been making the assumption that the kittens were interchangeable without even consciously making that choice. By thinking of the kittens as unique, I could then begin to understand Lucas's answer.

Shawn's explanation highlights an important aspect of subjective combinatorics: the role of unconscious assumptions in interpreting the problem. Initially, Shawn found Jamie's approach to the problem intuitive because it aligned with his own interpretation that kittens are identical. This assumption was made without explicit awareness. It was only after discussing the problem with a friend that Shawn recognized this underlying assumption and acknowledged that kittens could also be seen as unique. Shawn's experience underscores how unconscious biases towards viewing items as identical or unique can significantly influence students' combinatorial thinking.

Interpretation 2: Kittens as Unique

Real-world experiences. While all participants' responses suggest they agree people are inherently unique, some extend this perspective to kittens, recognizing their unique attributes.

Max: Jamie was assuming that the kittens were not unique. The kittens, like the children, are different, so Jamie's expression does not account for all the permutations of different children holding different kittens.

Max challenged Jamie's assumption by likening kittens to children, rejecting the idea that they can be considered identical, and emphasizing that each kitten, like each child, should be considered unique. This perspective is echoed by Heather, who supports Lucas's interpretation by emphasizing that kittens have unique markings, sizes, and personalities, much like humans.

Heather: I agree with Lucas. Kittens have both unique markings, sizes, and personalities therefore it is fair to treat them as unique objects. Humans are similarly typically considered unique objects. Therefore, we are ordering one set of unique objects onto another set of unique objects.

Heather extended this idea further by noting that the problem involves matching one set of unique objects (kittens) to another (humans), reinforcing the importance of recognizing these individual attributes in her combinatorial reasoning.

Matthew's explanation links real-world experiences and combinatorial thinking.

Matthew: When thinking about this question, Lucas appears to have an image of 10 distinct kittens that are up for adoption. This is a fair assumption, since each kitten would naturally be different. A typical animal shelter that offers kittens for adoption would have kittens that already have their own names, and a personalized description for each kitten before they are put up for adoption. During the process of acquiring a kitten, a person also usually chooses a kitten based on a number of factors such as their personality, gender, or compatibility with other animals, thus making each choice unique.

Matthew highlighted the realistic context of an animal shelter to emphasize the assumption of unique kittens. He drew a parallel between the counting problem and the real-world scenario of adopting kittens, where each kitten typically has a distinct name and description. This real-life context supports the idea that each kitten is perceived as unique, similar to how they are treated in actual adoption settings. Matthew's explanation illustrates how real-life contexts can shape students' assumptions in combinatorial reasoning.

Dual Interpretation: Kittens Could Be Either

Personal and mathematical connections. While the earlier discussion highlights the influence of real-life contexts on interpreting the uniqueness of kittens, the following excerpt explores how personal experiences in combinatorics and emotional connections can simultaneously lead to seeing kittens as both identical and unique. This dual interpretation highlights the complex interplay between combinatorial experiences and personal experiences in shaping one's approach to combinatorial problems.

Heather: Most math questions are set up so that if there are 10 kittens all must be distributed unless otherwise stated thus people tend to compare to other previous examples and will assume all kittens must be distributed. Also when discussing the adoption of kittens most people have an emotional preference to having them all adopted out. The confusion as to whether kittens are unique or not may either stem from a comparison to other math examples where objects are frequently treated as identical unless otherwise stated or from their experience with kittens and how they picture the scenario. For example, most people who have previously adopted a kitten and picture this scenario as 7 people going to an agency meeting the kittens and then going home with one or two kittens will know that going home with kitten 1 instead of kitten 2 could feel drastically different for that person. However, if the person is picturing a scenario where an adoption agency is just going to deliver kittens they have never met it might not matter which one you get since you never met kitten and kitten 1 is still a kitten you get to play with or perhaps the person has never met a kitten before and is unaware of their unique personalities, markings and sizes.

Heather's assumption that all kittens must be distributed and might be treated as identical stems from her experience with combinatorial problems. This reflects how her past experiences in combinatorics have shaped her initial subjective interpretation of the problem. However, she also noted that real-life experiences and emotional preference with kittens can influence students' interpretations, leading them to see the kittens as either identical or unique, depending on how they picture the scenario.

Level of detail. While Heather's explanation brings attention to how mathematical experience and personal experience both contribute to a dual interpretation, the following excerpt further explores how the level of detail and personal connection can shift one's interpretation.

Shawn: If I want to categorize something as "bird", I am probably only looking to see if it has a beak and wings. If I want to categorize something as a wood duck, I now have to consider the size, the colouration of the feathers, the body shape, and the habitat. The word kitten is a high level category. It can easily lead to thinking of kittens as generic and interchangeable. Whether one particular person thinks of them as interchangeable or unique probably depends on how they feel about cats in general, and whether those ten cats are physically in front of them. I initially approached the problem seeing the kittens as interchangeable, but if I had to choose a pet from a specific group of kittens, I would have a preference for which one I wanted, meaning that I felt that the kittens were unique.

Shawn's explanation touches on how the level of specificity in categorization can influence whether one perceives objects as interchangeable or unique. This suggests that how one frames the problem—whether focusing on broad categories or specific details—can shape the subjective interpretation of the elements involved. Shawn's shifting perspective—from seeing kittens as generic to recognizing their uniqueness based on situational factors—illustrates how subjective interpretations can evolve as more context or personal connection is applied. Similarly, the following excerpt contrasts how different levels of detail in viewing the kittens can lead to different interpretations.

Dmytro: If Lucas views kittens as complex with different colours, personalities, genders, etc. then in his eyes kittens aren't just kittens. Each one is complex enough in that they should be distinguished from one another. Where Jamie may view kittens as just kittens. More on a macro level like they are all just one species.

Dmytro's explanation highlights the varying levels of complexity with which individuals might perceive and categorize objects, such as kittens, in combinatorial problems. His distinction between viewing kittens as a homogeneous group versus recognizing their unique attributes underscores how subjective judgment influences whether objects are seen as interchangeable or unique. This analysis suggests that the level of detail one applies to a problem can significantly shift the outcome of their reasoning.

Discussion and Conclusion

In this paper, I introduce *subjective combinatorics*, with the goal of providing language to describe different interpretations involved in students' combinatorial thinking. As discussed in the introduction, in recent years, the mathematics education research community has increasingly focused on addressing students' underlying conceptualizations of combinatorial ideas. This study addresses subjectivity in students' combinatorial thinking, thereby filling a gap in the mathematics education research on the area of combinatorics.

To explore subjective combinatorics, I examined how participants attended to a counting problem that allowed for distinct interpretations. The study focused on two main aspects: the restrictions participants generated while engaging with counting problems and the impact of their interpretations on their combinatorial thinking. In particular, this study addressed the following research questions: *How does subjectivity affect students' combinatorial thinking? How do students generate restrictions as they solve and evaluate counting problems that are susceptible to be considered subjectively?*

In addressing the research questions participants were presented with a counting problem which highlighted students' disagreements regarding solutions, drew participants into a situation where they faced varied interpretations. The task encouraged participants to engage with the numerical expressions/formulas component of Lockwood's model. The variation in interpretations—whether treating kittens as identical or distinct—affected how participants made sense of the suggested answers. This variation demonstrated how different conceptualizations of the problem influenced their counting processes and considerations of sets of outcomes. The results highlight how different conceptualizations of the problem influence students' combinatorial thinking. Furthermore, the results indicate that participants' interpretations of the set of outcomes were significantly influenced by unconscious assumptions, real-world experiences, personal and mathematical connections, and the level of detail considered. The findings illustrate how these factors shaped students' views on the uniqueness or interchangeability of objects in combinatorial problems. This underscores the importance of considering subjective factors when analyzing students' counting activities, as these factors can lead to diverse and sometimes conflicting interpretations of the same problem.

This study reveals the subjective nature of students' combinatorial thinking and the significant impact of personal context on their interpretations. Recognizing how these subjective factors affect students' combinatorial reasoning can offer valuable insights for educators. This understanding can lead to more effective teaching strategies that consider the diverse and personal ways students engage with counting problems. Lastly, the focus on a specific type of combinatorial problem may not capture all aspects of subjectivity in students' combinatorial thinking. Future research could benefit from exploring subjectivity across a broader range of combinatorial problems to gain deeper insights into students' combinatorial thinking. Additionally, future research might explore approaches to formulating combinatorial problems that reduce subjectivity, potentially leading to more consistent interpretations of students' combinatorial thinking.

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