Characterization of Student Learning in a Transition-to-Proof Course

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This research examines the ways students in a transition-to-proof class learn what a mathematical proof is and how to read, construct, and write proofs. We interviewed four university students who had successfully completed a transition-to-proof course about what experiences they thought most influenced their learning about proofs. We then interviewed two instructors of the course about their goals for student learning about the nature of proof and how they devised their instruction to advance those goals. We first describe the activities students engaged in and that the instructors provided for students that supported their apprenticeship into proof-based mathematics. We further analyze how students' characterization of these activities influenced their views of proof.

Keywords: transition-to-proof, cognitive apprenticeship, abstract algebra

The National Council for Teachers of Mathematics includes reasoning and proof as one of five process standards that should be present in curriculum for K-12 students, suggesting that students learn to select and execute a variety of proof structures (NCTM, 2000). Undergraduate students majoring in mathematics or mathematics education must become proficient in both reading and writing proofs in preparation for careers in research and/or education. This proficiency is accompanied by an understanding of what constitutes mathematical proof. This study seeks to add to the growing body of literature on proof by examining the ways students learn what a mathematical proof is and how to read and write proofs. Four university students who had successfully completed a transition-to-proof course and two instructors of the course were interviewed in an effort to answer the following research questions:

1. How do students characterize their learning about what constitutes a mathematical proof in a transition-to-proofs course on abstract algebra?
2. How do instructors characterize student learning about what constitutes a mathematical proof in such a course and adjust instruction to facilitate that learning?

Literature Review

Stylianides proposed a three-part definition of proof that relies on the mathematical community within which a proof is considered (2007). A proof is an argument with the following characteristics: (a) the argument uses statements that are accepted as true by the community; (b) the argument relies on modes of argumentation that are accepted by the community; (c) the argument is represented in a way that is understood and appropriate for the community. We adopt this definition of proof because it attends to varying expectations for the presentation and acceptance of proofs in different communities. It is critical to understand the norms established within the community from which a proof originates, especially for students transitioning from courses that previously focused primarily on computation, modeling, or other forms of mathematical activity. Instructors of the same course at the same university may even have differing goals and beliefs about norms that should be adopted by students, but as suggested by Dawkins & Weber (2017) we anticipate the values that underly those norms are similar.

Universities with mathematics programs often designate a course supporting students' transition into proof-centered mathematics courses. These transition-to-proof (TTP) courses teach logic rules, proof structures, and set theory (David & Zazkis, 2018). Some TTP courses study specific topics in mathematics (e.g., algebra or analysis) after completing several units about standard proof techniques. A smaller percentage of universities choose to embed all proof instruction within a content area, most frequently analysis.

Theoretical Framework

We consider proof to be a social language, which Bakhtin characterized as "a way of speaking characteristic of a particular group in a particular sociocultural setting (Wertsch, 1991, pp. 95)." One method through which individuals learn a social language is through participation in a *community of practice,* a group of individuals dedicated to refining their practice and training new members (Wenger, 1998). Through interactions with more experienced members, the student engages in a *cognitive apprenticeship of* social expectations, heuristic strategies, and efficiency techniques (Collins & Kapur, 2022).

We adopt Vygotsky's *zone of proximal development* (ZPD) as a theoretical framework for data analysis of sociomathematical development in this study. The standard interpretation of the ZPD is the interplay between what an individual can do on their own and what they can do with the help of a more knowledgeable other (Vygotsky, 1987, pp. 208-209). Vygotsky stresses that all cognitive functions first occur between individuals in a social context (inter-psychological). As an individual becomes more comfortable and knowledgeable about the function, it moves to a psychological context (intra-psychological) (Wertsch, 1991, pp. 88-89).

A complementary framing of the ZPD is based on the interplay of development of spontaneous and scientific concepts (Vygotsky, 1987, pp. 214-222). *Spontaneous concepts* develop through everyday experience, usually without the aid of formal instruction. Ideas within a spontaneous concept are specific and lack conscious awareness, volitional control, and generalization. *Scientific concepts* are those learned through mediated interaction with their meanings, such as instruction. They are more general and systematized but lack the accessibility and applicability of spontaneous concepts. As an individual develops both spontaneous and scientific concepts, the complementary strengths of each support the development of the other.

Methods

Undergraduate and graduate research subjects were recruited for interviews after their successful completion of an Introduction to Abstract Algebra course. This course serves as the transition-to-proof course for the large southwestern university where the study took place. We recruited students who had successfully completed the course within the previous 2-3 years. Four students participated in the research study: Alex, Carly, Dana, and Bennett (pseudonyms). Alex (undergraduate) completed the course 7 months before the interview. She had not taken any other mathematical proof courses. Carly (undergraduate) also completed the course 7 months before the interview. Carly had been enrolled in the same course during a prior semester but withdrew after 10 weeks due to personal reasons. Since her successful completion of the course, Carly had taken introduction to analysis. Dana (graduate) completed the course 7 months before the interview. She took the courses concurrently with introduction to analysis and had successfully completed intermediate analysis at the time of the interview. Bennett (undergraduate) completed the course 19 months before the interview. He had since taken introduction to analysis, two advanced calculus courses, and cryptography.

Each research subject completed one interview lasting 45-60 minutes. Interviews were conducted and recorded via Zoom. Each student interview was divided into 4 parts.

Part 1: Students were asked questions to determine their general understanding of the TTP course's goals as well as specific moments in the course that provoked learning.
Part 2 : Students discussed noteworthy features of two proofs, the first provided by an instructor of the course and the second by the student as an exemplar of their work in the class.
Part 3: Students answered questions about the general nature and purpose of proof, including the context of a proof and the potential evolution of their understanding of proofs over time.
Part 4: The researcher explained the research questions to the student and asked them to describe the process of learning what a proof is and how to read and write proofs.

Instructor interviews contained parallel versions of parts 1, 2, and 4 of the student interviews. Dr. King & Dr. Williams had both recently taught Introduction to Abstract Algebra. Dr. King is an experienced professor who finished teaching the TTP for the first time two months before the interview. Dr. Williams is an assistant professor at the beginning of her career.

All interviews were transcribed for analysis. We reviewed each transcript multiple times, searching for commonalities and differences between student learning experiences. We conducted open and axial coding to identify common themes in the data (Corbin & Strauss, 1990).

Results

Throughout the interviews, students described activities that positively impacted their ability to learn what proof is and how to read and write proofs. The instructors described ways they believed students learn proof and how they facilitated such activities. Each type of activity is described below with examples of how they specifically impacted students' learning of proof.

Conversing with a More Knowledgeable Other.

All four students reported talking to another individual about specific proofs while they were taking the course. In most cases, the more knowledgeable other was the instructor or a student who had been successful in the class before. Alex and Bennett frequently visited their professor's office hours where they had meaningful conversations about homework problems. Bennett claimed that office hours were extremely helpful if an individual prepared for them beforehand. He reported attempting homework problems and going to office hours to discuss specific lines in the proof where he was unable to continue. Bennett's professor tailored his conversations to Bennett's current conception of proof during office hours. The ensuing conversations helped Bennett complete missing pieces of his proof. Carly formed a small study group with another student enrolled in the course. She claimed that simply verbalizing her proof ideas and drafts to her friends often helped push her to more proficient proof production. Carly would put an outline of her proof on a whiteboard and then verbally explain the connections she was going to make throughout the proof. The act of sharing her proof draft with a peer highlighted the importance of making valid mathematical connections in a proof. Carly's social interaction with her peer supported her psychological development of proof writing techniques.

Attending to Written Feedback from the Professor

The students, particularly the three undergraduates, described the utility of feedback from their professor. Most referred to feedback on their homework, which included both a score and comments about the student's work. The inter-psychological experience of obtaining feedback helped support the intra-psychological development of proof concepts. Bennet contrasted the feedback he received in his first proofs course to feedback in other computation-heavy mathematics courses; he interpreted feedback from non-proof classes as an indication that he was doing something incorrectly. In contrast, he described feedback in his transition-to-proofs class as "the best thing you can be getting." Bennett described the struggle he had adopting the language norms of proof-writing, such as specific sentence structure or vocabulary. He felt that reading instructor feedback taught him many of the linguistic expectations of proof writing.

Carly reported that the instructor of her first transition-to-proofs course provided very little feedback. The second time she took the course, the feedback included edits to her proofs, such as replacing one word with another, which helped Carly understand the language norms of proof writing. If the feedback was unclear, she would speak with the professor to make sure she understood the instructor's intent. Dana claimed that both written and verbal feedback from her professor helped change her mental image of proof.

Dr. King provided extremely detailed feedback to students via Canvas, personalized to each student. He contrasted this with his Calculus courses, where he finds students typically making the same small set of mistakes. The idiosyncratic proof-writing of students in Dr. King's proof course required him to provide feedback that was more detailed repeatedly throughout the semester. His feedback often focused on components missing from student work, such as the presence of a predicate or a variable, and he noted that his students incorporated the feedback to write better proofs by the end of the course.

Consistent Class Attendance

Each of the four students commented about the importance of attending class regularly. Carly took rigorous notes that included the professor's board work and sidenotes referring to the context of classroom discussions. She also reported asking clarifying questions regularly throughout class and claimed that clear instruction and added logical commentary significantly improved her learning experience. Dana learned the value of scratch work alongside a written proof and found it particularly useful when her professor would model such aspects of proof development in class. This instruction helped her shift from memorizing and re-producing a select number of proofs to more engaged problem-solving.

Dr. King claimed that his general approach to instruction in a TTP course was the same as it was in a Calculus course. He discussed familiar concepts, taught them a new concept, and facilitated opportunities for students to experience the new concept productively. In other words, he reminded students of spontaneous concepts, taught a new scientific concept, and then supported the unification of the two. Before engaging in logical mathematical arguments, Dr. King engaged students in considering the logical nature of everyday statements, including using logic to determine the truth value of a statement. He hoped that analyzing non-mathematical statements in a formal, logical way would prepare students to analyze mathematical statements logically.

Exposure to Many Proofs

Each student talked about interacting with many different proofs. This interaction took on a variety of forms including example proofs from instructional materials, proofs found online, and proofs being studied in another course. Alex predominantly worked on her homework independently and in office hours. While working independently she found that reading proofs on the same topic improved her ability to recognize the utility of specific theorems or algorithms. She would search for proofs online that used the theorems she was studying in class. This helped her see how the theorem could be applied in the context of a proof. The perusal of proofs from online sources also provided her with the opportunity to repeatedly see what a proof looks like and how a proof is structured. She often tried to mimic the structure of proofs found online in her own homework assignments. In preparation for exams, Alex would attempt to prove a statement she had previously proved successfully. She would then compare her proof to one she had previously written, searching for specific missing components in her newly constructed proof. Dana found it challenging when the concepts in her analysis course assumed understanding from the transition-to-proof course she was concurrently enrolled in, but also found it helpful to practice her logic and proof writing skills consistently.

Dr. King felt students needed exposure to many proofs to refine their proof skills. In preparation for exams, he provided extensive study guides with 40-50 problems. His problems were designed to expose students to many types of proof within a wide range of categories, such as contrapositive proofs and proofs by cases. Dr. King said that successful students often worked through these problems and discussed them with him.

Discussion

The activities described above represent ways students are apprenticed into proof-based mathematics. The personalized nature of instructor feedback to students helped students refine their conceptions of proof to more closely align with the norms upheld by the instructor. As instructors made small corrections to student language and mathematical claims, students came to understand the mathematics of proof and the appropriate ways to communicate. Instructors' identification of specific proof components missing in a student proof reinforced the need to attend to a new set of norms of criteria for a valid solution.

Modelling and mentoring played a pivotal role in these students learning of proof. They benefited from instructors modeling not only the structure of a final product but also the problem-solving that accompanies such a product. As students adopted these problem-solving techniques, they grew in their ability to recognize a proof and produce one of their own. Proofs students encountered in other settings, such as proofs found online, also acted as models for what constitutes a proof.

Both students and instructors were aware of the differing needs of students in TTP courses. The needs of students in the TTP course were greater and different in nature than those of students in a computation-heavy courses. Instructors facilitated opportunities for students to converse with more knowledgeable others through office hours, study groups, and written feedback. Instructors were also mindful of the quality and the quantity of proofs students experienced as part of the coursework.

As we continue this vein of research, we will consider data collection methods that investigate social interactions between the student and others, including the instructor and peers. It is perhaps easy to anticipate the positive effects of frequently conversing with the professor of the course. However, it seems that for some students a conversation with a novice may be equally beneficial as a conversation with an expert. When conversing with another novice, the student is perhaps more likely to articulate basic reasoning, which can lead to more concrete formulations of mathematical concepts. Analysis of these critical social interactions will be included in future studies.

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