A Gendered-Language Analysis of Abstract Algebra, Real Analysis, and Introduction to Proof Textbooks

Kathleen Melhuish
Texas State University

Elizabeth Wrightsman
Texas State University

Lino Guajardo
Texas State University

Mario Gonzalez
Texas State University

Keith Weber
Rutgers University

Kristen Lew
Texas State University

Mathematics education scholars have asserted that proof can be viewed as a unique genre of communication with specific norms and conventions. Further, some have argued that these norms and conventions are gendered and may dissuade women from participating in advanced mathematics. However, these assertions for the most part have not been investigated empirically. In this proposal, we systematically analyze the linguistic features of 90 proofs from 15 textbooks in advanced undergraduate mathematics. Key findings include that many feminine ways of communication rarely occur in proofs, including hedges, references to emotions, and intensive adverbs. Some masculine ways of communication, including the use of directives, are common in undergraduate proofs. Implications of this research are discussed.

Keywords: Gender; Genre; Linguistics; Proof

Mathematical proof plays a vital role in advanced mathematics. Proof is the dominant means of justification in advanced undergraduate mathematics classes and testing students' ability to prove statements is a primary means of assessing students' performance in those courses. For this reason, proof serves as a gatekeeper to studying advanced mathematics and participating in the community of professional mathematicians. Students who do not master the practice of proving will neither succeed in their advanced undergraduate mathematics courses nor be invited to study mathematics further.

There are several reasons for why reading and writing proofs is challenging for university mathematics students (see Stylianides et al., 2017, for a review of this literature). In this paper, we focus on one particular source of difficulty: to read and write proofs, undergraduate mathematics students must work with a new genre of academic writing. In recent years, several scholars have remarked that proof can be viewed as rhetorical genre (Bowers & Küchle, 2020; Lew & Mejía-Ramos, 2019; Selden & Selden, 2013). There are a number of norms and conventions in proof writing that seem unique to this genre and are not present in most other forms of discourse. For instance, mathematics educators claim that proofs usually do not use the "I" pronoun (Burton & Morgan, 2000; Dawkins & Weber, 2017). Lew and Mejía-Ramos (2019) demonstrated that students are not aware of these norms, and they speculate that this may contribute to students' difficulties in engaging with proof. We note, however, that the norms and conventions of proof described in the mathematics education literature are usually not based on empirical studies on how proof are written. Rather, authors commonly justify their claims by referencing their own experience, or by citing other scholars who have made similar assertions. One contribution of this report is to provide an empirical study exploring trends on how 90 proofs are written in undergraduate textbooks.

Scholars have further asserted that the norms and conventions of proving may lead to inequities and discourage students from underrepresented populations from participating in advanced mathematics (e.g., Brown, 2018; Tanswell & Rittberg, 2020; Minggi, et al., 2021). In

particular, some scholars claim that proof may represent a masculinized way of communicating (Keith, 1988; Mendick, 2006), since proofs are impersonal and directive. If true, this can dissuade women from participating in advanced mathematics, as there is a larger gap between the norms of proof writing and the way that women would ordinarily communicate in natural settings (c.f., Tanswell & Rittberg, 2020). However, claims that proof is indeed a masculine form of communication are in need of empirical support. In this paper, we investigate this hypothesis by analyzing 90 proofs from undergraduate textbooks.

Related Literature and Framing

Like Selden and Selden (2013) and Bowers and Küchle (2020), Dawkins and Weber (2017) assert that proof is a genre where proofs are expected to adhere to specific norms. Dawkins and Weber argue that these norms are not arbitrary. Rather, these norms are in place for epistemological reasons. These norms support axiological virtues of the discipline of mathematics and facilitate the epistemological aims of the intended audience for these proofs.

Dawkins and Weber (2017) first assert that mathematicians desire that mathematical truth be *a priori* and objective, corroborating Ernest's (1995) claim that mathematicians have absolutists views where their knowledge claims are correct, universal, and incorrigible since they flow from firm foundations via deductive logic. Two norms associated with this value are that justifications must be traced to stipulated definitions, and that justification should be deductive and not allow for rebuttals (c.f., Inglis et al., 2007). This leads proofs to be written as chains of deductions. As Pedemonte (2007) summarizes: "[W]ithin proofs, the steps are connected by a 'recycling process' (Duval 1992–1993): the conclusion of one step serves as an input condition to the next step" (p. 24).

Dawkins and Weber (2017) further argue that "mathematical knowledge and justification should be independent of (non-mathematical) contexts, including time and author" (p. 129). This points to a common refrain that proofs aim to conceal the author and render them without a human element (e.g., Burton & Morgan, 2000). Further, proofs are "addressed to a universal audience" (Pedemonte, 2007, p. 27). This leads to the norm that references to the personal experiences of the author are omitted, including the process used to generate the proof and even the pronoun "I".

A final epistemological virtue mentioned by Dawkins and Weber (2017) is that proofs should be written to enhance understanding. Proofs are written in a brief and clear style so that mathematicians can recognize the overarching structure of the proof and recognize their key ideas. To avoid getting bogged down in logical minutia, routine details from a proof are omitted, and the proofs do not contain redundancy (Selden & Selden, 2013). We note here that this sparse writing style might not actually increase comprehensibility, especially for students. Logically compact statements can be difficult to parse, and "routine" details might not be obvious for a student to supply (see Minggi et al., 2021).

Gendered Communication

Women and men tend to communicate differently. In Table 1, we present Mulac et al.'s (2001) synthesis of the documented significant differences between male and female communication. As can be seen from Table 1, the literature reveals that men are more likely to use directives and judgmental adjectives in their communication and women are more likely to express uncertainty, add hedges, or use intensive adverbs. With Table 1 in mind, our research questions are: (R1) To what extent do proofs in advanced undergraduate mathematics textbooks

contain the 18 language features described in Table 1 and (R2) Which masculine features are common in proofs in advanced undergraduate mathematics textbooks, and which feminine features are absent?

Table 1. Features of Language Found in Empirical Studies (Mulac et al., 2001) where M: male features, F: female features; X: mixed results from empirical studies.

Language Variable [1]	Definition
Questions (X)	But not including directives in question form
Tag questions (F)	An assertion that is followed by a question asking for support
Directives (M)	Telling another person what to do
Negations (F)	A statement of what something is not
Sentence initial adverbials (F)	Answers the questions: how, when, or where? regarding the main clause [why and how long]
Oppositions (F)	Retracting a statement and posing one with an opposite meaning
Judgmental adjectives (M)	Indicating personal evaluations rather than merely description
Uncertainty verbs (F)	Verb phrases indicating lack of certainty or assuredness
Progressive verbs (X)	Verbs presented in the "-ing" form
Intensive adverbs (F)	Adverbs such as "really" or "so"
Hedges (F)	Modifiers that indicate lack of confidence in, or diminished assuredness of, the statement
Justifiers (X)	A reason is given for a previous statement
References to emotion (F)	Any mention of an emotion or feeling
References to quantity (M)	References to an amount or quantity
Locatives (M)	Usually indicating the location or position of objects
Personal pronouns (X)	Words that stand for beings
"I" references (M)	First person singular pronoun in the subjective case
Fillers (X)	Words or phrases used without apparent semantic intent

1 We do not report on two categories from the framework: elliptical sentences (due to lack of consensus in coder team) and dependent clauses (due to their common appearance across all types of texts Rafajlovičová, 2013).

To avoid misinterpretation, we wish to emphasize two points. First, Table 1 represents broad trends in the differences between how men or boys and women or girls communicate, but these are only statistical trends, and do not tell us how a particular individual will communicate. There are doubtlessly specific females who will frequently use directives and specific males who will avoid them. Second, Table 1 does not specify *why* males and females communicate differently. We agree with Mulac et al. (2001), gender can be understood as a culture in which communication is learned and practiced. That is, "much of what people know about interpersonal communication is learned from same-sex peers during the ages of 5 to 15 years," and therefore particular cultural elements, including language and communication, are developed within same-sex groups. Of course, power, societal ideologies, and ambient cultural trends likely shape how children learn to communicate in their formative years. We also note that this empirical synthesis only allows for exploration related to binary gender identities.

Methods

To explore the nature of language in proof textbooks, we selected the five most used books at the undergraduate level for each of: Introduction to Proof, Abstract Algebra, and Analysis. To identify these books, we used the analysis by David & Zazkis (2020) for Introduction to Proof, Melhuish (2015) for Abstract Algebra, and use the Open Syllabus (2023) project to identify Analysis textbooks. Based on a power analysis (80%) to capture a 0.5 effect size (f) with 15 groups (reflecting 15 textbooks), a sample size of 90 was needed to detect medium differences between books. In this paper, we focus primarily on overarching trends, but this will allow for more detailed analysis in the future.

We selected six proofs per textbook. We selected these six proofs as follows. For each textbook, we set n equal to the total page count divided by 6. We went to page n, and proceeding backwards, identified the first proof. We then repeated this process for $2n$, $3n$, $4n$, $5n$, and $6n$, thus collecting six proofs that span the textbook. Each proof was then coded using the linguistic features in Table 1. This began with a common coding of eight proofs in a pilot to develop a shared understanding of codes. After initial coding, the team met to discuss questions, inconsistencies, and to come to a consensus on how each code was applied. At this point, a revised primary coding was finalized for each proof and is reported below.

Results

Many of the linguistic features we analyzed were consistent across proofs. Table 2 provides an overview of the frequencies of the different features.

Table 2. Frequencies of Features in Proofs Analysis where) where M: male features, F: female features; X: mixed results from empirical studies (Mulac et al., 2001).

Language Variable	Proofs with This Feature (n=90)	Instances of This Feature
Questions (X)	0	0
Tag questions (F)	0	0
Directives (M)	78	209
Negations (F)	26	54
Sentence initial adverbials (F)	90	626
Oppositions (F)	4	5
Judgmental adjectives (M)	23	40
Uncertainty verbs (F)	2	2
Progressive verbs (X)	38	83
Intensive adverbs (F)	10	13
Hedges (F)	3	3
Justifiers (X)	27	41
References to emotion (F)	1	1
References to quantity (M)	46	198
Locatives (M)	18	46
Personal pronouns (X)	76	323
“I” references (M)	0	0
Fillers (X)	0	0

To organize the remainder of the results discussion, we suggest three distinct types of features: necessary, arbitrary, and epistemological. We borrow the distinction between arbitrary

and necessary from Hewitt (1999) but delineate between the two slightly differently. A necessary feature is one necessitated by the mathematical content involved in the proof. We then make a distinction between two types of non-necessary features: arbitrary and epistemological. An epistemological feature is one rooted in the way knowledge is established by the mathematical community. We use arbitrary feature for one that does not link to a particular content need or means of knowledge generation. Both epistemological and arbitrary features are reflective of convention, but as noted by Kontorovich and Zazkis (2017) convention choices are meaningful in the discipline.

Necessary Content-Based Linguistic Features

Some linguistics in proof are what we refer to as *content-based features*. Many proofs had references to quantity (n=46) and locatives (n=18) , which seem unavoidable in mathematical domains that concern number and space. Arguably, negations are a content-based feature as well. The negations that we observed in the proofs for the most part concerned proofs by contradiction.

Epistemological Linguistic Features

In many cases, we could relate the presence or absence of some linguistic features to epistemological values such as those described in Dawkins and Weber (2017). We refer to these linguistic features as *epistemological features.* For instance, across all 90 proofs, there were three instances of hedges, two instances of uncertainty verbs, and no tag questions. Each of these three linguistic features was described as more common in feminine communication. The absence of these linguistic features is consistent with the view that proofs should provide certainty. There was only one reference to emotion across the entire data set, which also has been found to be a feminine way of communicating. This finding is consistent with the content of proofs being independent of the author or the reader (in particular, the emotional reactions or states of the authors or readers).

The use of the "I" pronoun is found to be a masculine way of speaking. Both Burton and Morgan (2000) and Dawkins and Weber (2017) assert that the "I" pronoun is rarely used in proofs, with Dawkins and Weber (2017) speculating that this norm is in place so that proofs are independent of their author. We found support for the norm to avoid the "I" pronoun; none of the 90 proofs used this pronoun. The "we" pronoun was used frequently, accounting for all 323 personal pronouns in the proofs. Although the use of personal pronouns was not distinctly masculine or feminine, we found that the "we" pronoun was often used to give directives (e.g., "We observe…"), and as a "royal we". So, this may still nonetheless may represent a depersonalized and masculine form of communication (see Weber, 2023, for more discussion). Interestingly, Hungerford (2012) did not use the word "we" in any of the six proofs that we analyzed, suggesting that the use of "we" may be a stylistic choice by the author.

In contrast, we were surprised to see judgmental adjectives (a masculine way of communicating) were present in 23 of the 90 proofs that we analyzed with statements such as "we merely have to show..." or "it is obvious that…" This would seem to go against upholding the value that proofs are independent of the authors who wrote them.

Arguably other features that may align with the values of the discipline and knowledge generation include the frequency of "sentence initial adverbials" (found in every proof on average 7 times) and "justifiers" (found in 27 proofs, 41 times). Justifiers provide a reason for a fact after it is presented. These were unexpectedly low. Sentence initial adverbials serve to connect a new state from the prior. The justification in proof is often found *before* a statement,

and thus not meeting the justifier definition. That is, these justifications are found in the initial sentence adverbials which often look like "Thus" or "By theorem X." We note that these features align with the deductive logic that underscores proofs. Each line follows deductively from the previous illustrating the recycling process suggested by Pedemonte (2007).

We also note progressive verbs were not found commonly amongst the proofs; another feature found more often in how women communicate. Thirty-eight of the 90 proofs contained progressive verbs. These are action verbs that end in "ing." We suggest this again aligns with epistemological features as truth is not established via the problem-solving process (Dawkins & Weber, 2017) and therefore active verbs would be uncommon in the final product.

Arbitrary Linguistic Features

Some of the linguistic trends that we observed appeared to us to be independent of the content being studied and did not appear to support an epistemological value. We refer to such features as arbitrary linguistic features. Intensive adverbs, a feminine form of communication, were infrequent in the proofs that we surveyed. We found only 13 intensive adverbs appearing in 10 proofs. We suggest that proofs tend towards brevity and words that do not serve to provide additional meaning are not commonly found. We also suspect this desire for brevity may account for the breach of the objective value because statements like determining something is "obvious" often serve the purpose of reducing the amount of information needed to be shared.

Directives, a masculine form of communication, were common in the proofs that we observed. A total of 209 directives appeared in 78 of the 90 proofs that we surveyed. We discuss directives as an arbitrary feature because they do not necessarily connect to the deductive argument development of knowledge. Sometimes they serve the role of introducing notation (e.g., "let x be a...") while other times serve to draw attention to information (e.g., "note ..."). Directives are one of the few features that have been empirically studied. Inglis and Tanswell (in press) found that imperatives where common in mathematicians' proofs on the mathematical archive, and Weber (2023) illustrated how directives (which he labeled "instructions") were frequently used in graduate set theory textbooks. Our results corroborate these findings in undergraduate mathematics textbooks.

Discussion

A number of mathematics educators have speculated on the nature of the genre of proof (e.g., Bowers & Küchle, 2020; Selden & Selden, 2013). However, empirical analysis of the linguistic features of proofs in advanced undergraduate mathematics courses is uncommon. Table 2 reveals several trends based on 90 proofs in 15 textbooks that we analyzed. In many cases, our results corroborated the claims made by other scholars, such as proofs rarely use the "I" pronoun. In other cases, we documented (perhaps not surprising) trends in proofs, such as the lack of hedges, uncertainty adverbs, and intensive adverbs. Finally, there was the curious finding that judgmental adjectives appeared in 23 proofs. This warrants further analysis.

Our findings also indicate several linguistic features of proofs that are representative of masculine ways of communicating. These include the lack of tag questions, uncertainty verbs, hedges, intensive adverbs, and references to emotion, the frequent use of directives, references to number, and locatives, and the occasional use of judgmental adjectives. We note that not every masculine linguistic feature of communication from the literature was common in the proofs that we analyzed. In particular, the pronoun "I", a masculine form of communication, was not present in any of the 90 proofs that we coded.

This raises the question, do proofs need to contain masculine linguistic features and avoid feminine linguistic features? Our current analysis is speculative. It amounts to us asking whether proofs need to have the linguistic features that they do, and can we imagine presenting proofs of the same content in different ways. We cannot see how the content-based linguistic features that proofs have can change. Much of mathematics concerns numbers, quantities, and location, and as such, proofs will often have references to quantity and locatives. It is easier to imagine altering the arbitrary linguistic features. For instance, one can add intensive adverbs to a proof without changing its content, and one can write proofs in a way that does not direct the reader to perform actions. What we coded as epistemological linguistic features present a more interesting case. It is true that avoiding reference to emotion may uphold the value that proofs are independent of author, and avoiding hedges and uncertainty may uphold the value that mathematical results can be known with certainty. However, as noted by Ernest (1995), these epistemological values themselves may be masculine. He suggests:

> There is a radical view that the kind of popular image of mathematics described above serves conservative interests in the mathematics community and in society in general. For if mathematics is viewed as difficult, cold, abstract, ultra-rational, important and largely masculine, then it offers access most easily to those who feel a sense of ownership of mathematics, of the associated values of western culture and of the educational system in general (p. 454).

He further references the work of Gillian (1982) which argues that women and men have differed values in their development where, women are more likely to have values that are, "relationships, connections, empathy, caring, feelings and intuition, and tends to holistic and human-centred" (p. 454). This is contrasted with a separated view where men are more likely to have values that emphasize, "rules, abstraction, objectification, impersonality, unfeelingness, dispassionate reason and analysis, and tends to be atomistic and thing-centred in focus" (p. 454). If this assertion is accurate, then the proof features aligning with epistemological values may need to be interrogated for the way they encode gendered and power dynamics. For example, one can imagine proof norms where the author is allowed to express an emotional state (surprise at a non-intuitive result, or frustration that a particular proof approach is difficult). Further, while proofs must employ deductive reasoning, we can imagine proofs where the author acknowledges that they are not absolutely certain that they carried out each deductive step correctly. Mathematicians offer these types of hedges in their ordinary discourse around proofs (Weber, Mejía-Ramos, & Volpe, 2022), so we can imagine permitting them to do so within the proof itself.

We have demonstrated that proofs in undergraduate mathematics frequently possess linguistic features that are associated with masculine forms of communication. Bowers and Küchle (2020) noted that genres have a power dimension and may play an exclusionary role. It is therefore reasonable to conjecture that the linguistic norms around proof writing may serve to exclude students (Keith, 1988). This suggests two lines of future research. The first is seeing how these linguistic norms encourage or discourage participation in the practice of proving, particularly in gendered ways. Would women, nonbinary individuals, (or even men) be more comfortable in writing proofs in ways that did not adhere to the masculine norms of proving? The second is more explorative and involves designing classrooms that re-imagine proof discourse.

References

Brown S (2018). E-IBL: An exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. In Weinberg, A., Rasmussen, C., Rabin, J., Wawro, M., and Brown, S. (eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*. 1–15.

Bowers, D. M., & Küchle, V. A. (2020). Mathematical proof and genre theory. *The Mathematical Intelligencer*, *42*(2), 48-55.

Burton, L., & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education*, *31*(4), 429-453.

David, E. J., & Zazkis, D. (2020). Characterizing introduction to proof courses: A survey of US R1 and R2 course syllabi. *International Journal of Mathematical Education in Science and Technology*, *51*(3), 388-404.

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics,* 95, 123-142.

Ernest, P. (1995). Values, gender and images of mathematics: A philosophical perspective. *International Journal of Mathematical Education in Science and Technology*, *26*(3), 449-462.

Gilligan, C. (1982). *In a different voice: Psychological theory and women's development.* Harvard University Press.

Gudykunst, W. B., & Ting-Toomey, S. (with Chua, E.) (1988). Culture and interpersonal communication. Newbury Park, CA: Sage.

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum. *For the learning of Mathematics*, *19*(3), 2-9.

Hungerford, T. W. (2012). Abstract Algebra: An Introduction. Cengage Learning.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*, 3-21.

Inglis, M., & Tanswell, F. S. (forthcoming). The language of proofs: A philosophical corpus linguistics study of instructions and imperatives in mathematical texts. In B. Sriraman (Ed.), *Handbook of the history and philosophy of mathematical practice*. Springer.

Keith, S. (1988, June). Women and communication in mathematics: One woman's viewpoint. Paper presented at the 10th Annual Meeting of the National Women's Studies Association, Minneapolis, MN.

Kontorovich, I., & Zazkis, R. (2017). Mathematical conventions: Revisiting arbitrary and necessary. *For the Learning of Mathematics*, *37*(1), 29-34.

Lew, K., & Mejía-Ramos, J. P. (2019). Linguistic conventions of mathematical proof writing at the undergraduate level: Mathematicians' and students' perspectives. *Journal for Research in Mathematics Education*, *50*(2), 121-155.

Lew, K., & Mejía Ramos, J. P. (2020). Linguistic conventions of mathematical proof writing across pedagogical contexts. *Educational Studies in Mathematics*, *103*(1), 43-62.

Maltz, D., & Borker, R. (1982). A cultural approach to male-female misunderstanding. Language and social identity, 198-215.

Melhuish, K. M. (2015). *The design and validation of a group theory concept inventory* (Doctoral dissertation, Portland State University).

Mendick, H. (2006). *Masculinities in mathematics*. McGraw-Hill Education (UK).

Minggi, 1, Alimuddin, T & Sabri, S. (2021). Language Constraints in Constructing Arguments for Mathematical Proofs. In *Journal of Physics: Conference Series* (Vol. 2123, No. 1, p. 012046). IOP Publishing.

Mulac, A., Braac, J. J., & Gibbons, P. (2001). Empirical support for the gender-as-culture hypothesis: An intercultural analysis of male/female language differences. *Human Communication Research*, *27*(1), 121-152.

Open Syllabus. (2023). https://www.opensyllabus.org/

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed?. *Educational studies in mathematics*, *66*(1), 23-41.

Rafajlovičová, R. (2013). Subordinate clauses–their forms and functions in different text types. *English Matters IV: A collection of papers by the Institute of British and American Studies*, 42-48.

Segal, J. (1999). Learning about mathematical proof: Conviction and validity. *The Journal of Mathematical Behavior*, 18(2), 191–210.

Selden, A., & Selden, J. (2013). The Genre of Proof. In K. Weber (Ed.), Reflections on Justification and Proof. In T. Dreyfus (Ed.), Mathematics & Mathematics Education: Searching for Common Ground (pp. 248–251). Dordrecht, the Netherlands: Springer.

Solomon, Y. (2006). Deficit or difference? The role of students' epistemologies of mathematics in their interactions with proof. Educational Studies in Mathematics, 61(3), 373–393.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 237–266). National Council of Teachers of Mathematics.

Tanswell, F. S., & Rittberg, C. J. (2020). Epistemic injustice in mathematics education. *ZDM*, *52*(6), 1199-1210.

Weber, K. (2023). Instructions and constructions in set theory proofs. *Synthese*, *202*(2), 34.

Weber, K., Mejía-Ramos, J. P., & Volpe, T. (2022). The relationship between proof and certainty in mathematical practice. *Journal for Research in Mathematics Education*, *53*(1), 65-84.

Weber, K., & Melhuish, K. (2022). Can we engage students in authentic mathematical activity while embracing critical pedagogy? A commentary on the tensions between disciplinary activity and critical education. *Canadian Journal of Science, Mathematics and Technology Education*, *22*(2), 305-314.