Towards a Unified Theory of Abstraction: Abstracting and Reducing Abstraction as Reciprocal Categories of Mathematical Practices

Alice Mehalek
Colorado State University

*Keywords:* Abstraction, Reducing Abstraction, Unified Theory, Abstract Algebra, Disciplinary Practices

In the undergraduate classroom, abstract algebra concepts can be presented at varying levels of abstraction. Traditional lecture-based instruction presents concepts abstractly, and students often rely on their own cognitive strategies to make sense of them (Hazzan, 1999). Instructors using inquiry-based teaching methods often begin at a lower level of abstraction via concrete examples, guiding students to a higher level of abstraction through activities by which they discover and formalize the abstract properties of algebraic structures (for example see Ernst, 2016; Larsen et al., 2013). Many courses use a combination of teaching methods, requiring students to navigate between higher and lower levels of abstraction in both directions.

Existing theories of abstraction usually address activities and/or cognitive practices in one direction. Hazzan's (1999) reducing abstraction framework, expanded by Melhuish et al. (2018), classifies cognitive strategies that abstract algebra students use to *reduce* abstraction level. Other frameworks in the literature, including the generalization framework (Ellis et al., 2017) and Advancing Mathematical Activity (Rasmussen et al., 2005), describe ways in which students' understanding of mathematical topics *increases* in abstraction level through activities (referred to as disciplinary practices) such as generalizing, defining, and symbolizing.

Drawing on the existing literature and informed by my observations, I propose a unified theory of abstraction that connects multiple theories of learning through their approaches to abstraction. This unified theory comes out of my own need to make sense of observations of students' abstracting activities within the context of my dissertation study investigating abstraction in a groups first versus a rings first abstract algebra course, in which I observed the students and instructor moving fluidly between more and less abstract representations of abstract algebra ideas. Thus, this theory positions *abstracting* and *reducing abstraction* as reciprocal sets of linked disciplinary practices. This positioning reflects the bidirectional nature of navigating between levels of abstraction and conjectures that fluency in both directions is an indicator of student learning.

*Abstracting* encompasses the practices by which students move from less abstract to more abstract, including practices such as decontextualizing, relating, extending, generalizing, and symbolizing. *Reducing abstraction* encompasses the practices by which students move from more abstract to less abstract, and includes practices such as contextualizing, exemplifying, computing, and embodying. I define these practices within five categories of abstraction. Three of the categories (relationship between the object of thought and the thinking person; reflection of the process-object duality; complexity of concept of thought) were defined in Hazzan's (1999) original framework and the fourth (degree of precision/formality) added in the expanded framework of Melhuish, et al. (2018). I add a fifth category, interpreting abstraction as the degree of concrete embodiment of the concept of thought, based on observations and the literature on concreteness fading (Fyfe et. al, 2014). I define some of the practices by which students abstract and reduce abstraction within each category and provide examples of how students use these practices from my observations of an abstract algebra class.

## References


Ellis, A., Tillema, E., Lockwood, E., & Moore, K. (2017). Generalization across domains: The relating-forming-extending generalization framework.

Ernst, D. C. (2016). An inquiry-based approach to abstract algebra. Retrieved on November 13, 2024.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational Psychology Review*, *26*, 9-25.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics*, *40*, 71-90.

Larsen, S., Johnson, E., & Weber, K. (2013). The teaching abstract algebra for understanding project: Designing and scaling up a curriculum innovation. *Journal of Mathematical Behavior, 32*(4), 691-790.

Melhuish, K., Bergman, A., & Czocher, J. (2018). Revisiting reducing abstraction in abstract algebra. In *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1197-1205).

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51-73.