Using Schemas to Model in APOS: The Case of the Differential Calculus of Two-Variable Functions

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In this report, we use the Schema portion of Action-Process-Object-Schema (APOS) theory to frame a study of student understanding of the differential calculus of two-variable functions. The use of Schemas allows us to obtain a more holistic perspective of students' understanding. Data was obtained from semi-structured interviews with eleven students. We argue that the APO portion of the theory can give detailed information about specific Schema components but that it does not inform about the interrelations between the different components. However, the notion of Schema and its instrumentalization via the notions of genetic decomposition, types of relations between components, and the triad stages of Schema development can provide a more comprehensive picture encompassing the different components of the differential calculus Schema. The notions of the scope of an instrument and the graph of a Schema are introduced to afford a visual representation of a student's Schema.

Keywords: APOS, schema, calculus, two-variable functions, differential multivariable calculus

APOS is a well-known cognitive theory that proposes that knowledge of a specific mathematical notion is constructed in stages of development, starting with the Action stage and going through the Process and Object stages of conceptualization. The different mental structures constructed in these stages can be organized in Schemas that evolve with time. When learning mathematics, students frequently meet complex situations in which different notions are interrelated to build a more developed structure. In these cases, the focus on single structures of the APO part of the theory is insufficient to capture the situation's complexity. In these cases, it is convenient to use Schemas to model understanding. Schemas provide a more holistic view of knowledge construction and development. In this article, we use the context of the differential calculus of two-variable functions (DC) to exemplify the use of Schemas and propose a way to represent graphically a student's DC Schema at a moment in time.

Literature review

Several early articles introduced APOS theory and its foundational ideas (e.g., Asiala et al., 1996; Dubinsky, 1991, 1994; Dubinsky & McDonald, 2001) with a more recent book (Arnon et al., 2014) summarizing the theory's development. The notion of Schema in APOS is developed from Piaget's ideas (1971, 1972). Since Piaget's ideas were developed in a context different from that of advanced mathematical thinking, they must be adapted to be useful in this new context. Different researchers have made their own adaptations, even for the context of mathematics education (e.g., Vergnaud, 1990), emphasizing different aspects. As with any theory, APOS grows with the contributions of the community of researchers that apply the theory to understand didactic phenomena. The need to further the application of Schemas to understand the didactics of the chain rule led a group of researchers to develop the notion of Schemas in APOS by introducing the idea of the triad of Schema development (Clark et al., 1997). Further development was spurred by attempting to discuss the complexity of a linear algebra course, leading to the notion of types of relations between Schema components (Trigueros, 2019).

Throughout the years, there has been research applying the APOS notion of Schema that deal with different notions, for example, calculus graphing Schema (Baker et al., 2000), implicit differentiation (Borji & Martínez-Planell, 2020), chain rule (Clark et al., 1997), limits (Cottrill et al., 1996), sequences (McDonald et al., 2000), two-variable function optimization (Martínez-Planell et al., 2023), differential equations (Trigueros, 2000), linear algebra (Trigueros, 2019), and modeling (Trigueros & Martínez-Planell, 2024).

The idea of a genetic decomposition for Schemas as a list of Schema components, relations between the components, and a description of the triad has been gaining traction, as may be seen in different publications (e.g., Martínez-Planell et al., 2023). This study's context is the differential calculus of two variable functions; for a literature review of the didactics of this context, we refer the reader to the survey in Martínez-Planell and Trigueros (2021). The article by Borji et al. (2023) is also helpful in this regard. It discusses different research studies on the differential calculus of two-variable functions underscoring their geometrical interpretation.

Theoretical Framework

Since APOS is a well-known theory, we only define what we need from the Schema portion of the theory. We refer the reader to Arnon et al. (2014) for definitions of Action, Process, and Object. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas, dealing with a specific mathematical notion or topic. A Schema is coherent in the sense that the individual can determine when a problem situation falls within the scope of the Schema. This definition of Schema suggests that they may be modeled with a list of components, a list of relations between Schema components, and a description of how the Schema may develop. This model is called a *genetic decomposition* (GD) of a Schema. A GD is neither unique nor meant to describe the best way a student may learn a particular notion. The notion of types of relations between Schema components, introduced by Trigueros (2019), has proved to be useful. A *correspondence* relation is established out of habit or repetition, but the student cannot justify it. It can result from repeatedly seeing two components used in tandem or from superficial comparisons of similarities between components. A *transformation* relation groups different components in a normative way, with perhaps one of them being used to justify or explain the relation between the others. The relation is used normatively in a given problem situation, although not necessarily so across different problem situations. A *conservation* relation is used normatively and fluently across different problem situations. Properties in one structure are conserved or translated to properties in other structures so that the structures seem to be interchangeably used whenever it is convenient. Types of relations are a tool to use when deciding the stage of Schema development of a student with respect to a particular notion N. A Schema is at the *Intra*-N stage of development when the components are isolated or mostly related by correspondence relations; at the *Inter*-N stage of development when there are almost no isolated components and transformation relations start to appear, grouping different components; at the *Trans*-N stage of development when almost all the components are interrelated by conservation relations. The use of the adjectives “mostly” and “almost” in the previous definition is an acceptance of the fact that a student might be in transition from one stage of Schema development to another.

Our research questions are:

How do we study students’ Schemas for the differential calculus of two-variable functions?

What didactical suggestions can we deduce from the study of students’ Schemas?

Methodology

Eleven engineering students spanning the range from above-average (3), average (5), to below-average (3), as chosen by the professor, agreed to participate in two semi-structured interviews two weeks after completing an introductory multivariable calculus course at a mid-tier Iranian university. The two interviews were held on different days, each lasting approximately 1 hour. Interviews were video and audio recorded, transcribed, individually analyzed by the researchers, and discussed among them until a consensus was reached. There were 17 problems in the interviews[1]. The students used activities[2] designed to help them do the constructions proposed in the revised GD described in Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). The activities used the ACE (Activities in class, Classroom discussion, Exercises as homework) pedagogical strategy (Arnon et al., 2014). In particular, students worked collaboratively in groups of three or four; each group turned in their collective work, which was promptly graded later; there were group presentations in which their answers were discussed; activities not finished in class were completed as homework; and common difficulties deduced were discussed in class. We do not detail the interview instrument or the activities since this report focuses on how Schemas were used to analyze the data.

While typically one starts research focusing on the Schema portion of the theory by proposing a GD for a Schema, this is not what happened in this study. A GD for the differential calculus of two-variable function (DC) using the APO portion of the theory had been previously proposed, tested with data from students, and revised by Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). The revised GD was also expressed in terms of the APO portion of the theory. This was followed by a second research cycle to test the revised GD. The data obtained in the second research cycle, when analyzed using the APO portion of the theory, allowed us to obtain information about some specific components of the DC Schema: partial derivatives (Borji et al., 2024), directional derivatives (Borji et al., 2023), and slope (Martínez-Planell et al., 2022). However, focusing on the APO portion of the theory did not allow us to discuss students' construction of relations between these components and other components like tangent plane, total differential, gradient, and two-variable function, which is what we intend to do in the present study. In this report, we look at the same data but from the perspective of Schemas.

To examine the existing data from the perspective of Schemas, we proposed a list of components and relations. We start by considering the main notions that were tested with the interview instrument as components: two-variable function (F), slope (S), tangent plane (TP), partial derivative (PD), directional derivative (DD), gradient (G), and total differential (TD). To propose the relations, we ask: What can the interview instrument reveal about the relations students establish between different notions of the differential calculus of two-variable functions? The idea of the scope of an instrument proposes one answer to this question. The scope of an instrument is the collection of relations that are directly explored by the instrument or that may be introduced by the student in answering the questions in the instrument. The scope of an instrument may be represented graphically as follows. Each interview question explores at least one such relation. For example, in question 3a, we gave students the graph of a two-variable function and asked about the sign of the partial derivative at a point. A function was given (graphically), and information about a partial derivative (its sign) was requested. So, the question directly explores a relation between the Schema components of two-variable function and partial derivative. We represent a relation that is directly explored by the interview instrument with a

[1] The instrument is at https://drive.google.com/file/d/1ebXyptqwq2BAYYgkq-s02ijmpVfYBtx3/view?usp=sharing.
[2] See activities at https://drive.google.com/file/d/1cef_UjVXt9KxQemVK7hDFBioNND4CfzZ/view?usp=sharing.

solid line, as in Figure 1 (left). Further, the same question allows us to uncover other relations that students may establish. For example, in the same question, a student may decide the sign of the partial derivative by arguing about the slope of the tangent line at the point and thus use the component of slope to mediate the answer. We represent relations that students may indirectly address when answering an interview question with thinner lines, as in Figure 1 (right).

The relations that are directly or may be indirectly explored by the interview instrument are represented in Figure 2. This figure allows us to visually assess how complete an exploration of relations between components an interview instrument can afford. In this case, we see that the interview instrument allows us to inquire into the construction of many of the possible relations in the Schema.

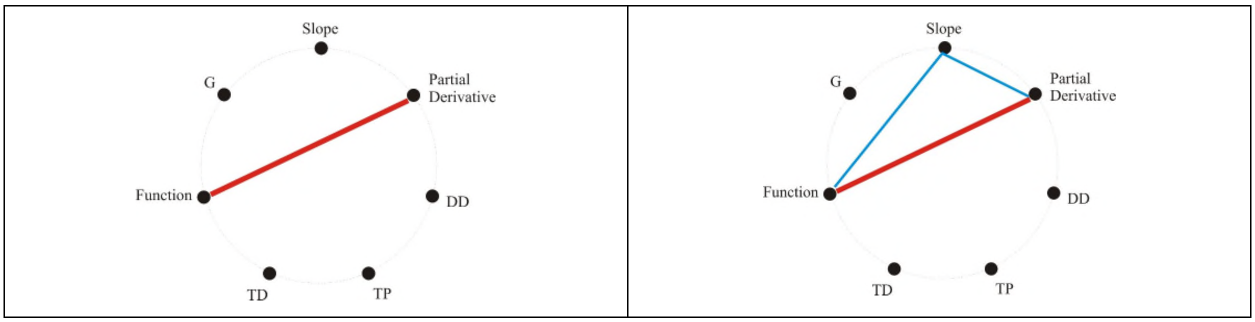


Fig. 1. Relation directly explored by interview question 3a (left) and relations that may be exhibited by a student when answering the same interview question (right).

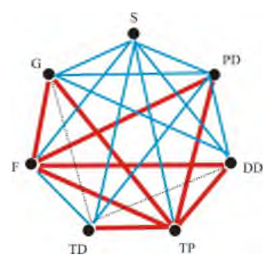


Fig. 2. Scope of the interview instrument. Relations that are directly (thicker lines) or indirectly (thinner lines) explored by the interview instrument. Dotted lines represent relations unlikely to be constructed by students.

After having analyzed the data using the APO portion of the theory, the data was analyzed again using codes that referred to the components and relations obtained from the interview instrument. For Schemas, we needed three distinct rounds of analysis. In the first round, we considered the relation established by the student in each question, if any, and classified its type. For example, if the student was given the graph of a two-variable function and asked for the sign of a specific partial derivative, we could use the codes F-S-PDc (the "c" is for correspondence), F-S-PDt (the "t" is for transformation), or F-S-PDe (the "e" is for equivalence or conservation) if the student talked about slope when answering the question, or perhaps F-PDc, F-PDt, or F-PDe if the student decided the sign based on properties of the function (increasing, decreasing) without mentioning slope. While the first round looked at the data question by question, the second round looked at it longitudinally. That is, we looked at all questions for a given student and verified that if an "e" had been used, the relation had been used in a normative way throughout the interview. We could also upgrade a relation initially classified as "t" to "e" if it had been used fluently and in a normative way throughout the entire interview. In the third round, we classified the students' Schema development as Intra-DC, Inter-DC, or Trans-DC. To do so, we used the students' "graph of Schema development" to obtain a visual representation of the Schema development. To construct such a graph, we draw line segments between nodes equally distributed in a circle (like in the graph for the scope of an instrument), but this time, the thickness of the line segment depends on the type of transformation (conservation-thick, transformation-thin, correspondence-dotted).

Results

We first analyzed students' construction of relations and the types of relations. We only include an example of one problem situation. For a more complete discussion of the difference between conservation, transformation, and correspondence relations, see Trigueros et al. (2023). Given the graph of $z = f(x, y)$ (Figure 3), students were asked for the sign of $f_{yx}(3,0)$.

Hamid: This second partial derivative is the derivative respect to y and then respect to x. So I need to compare the slope of tangent lines in the y direction while we move in the x direction… Okay. Before the point (3,0) I consider a point like this, umm the tangent line to the curve of the surface in the y direction is a line like this and its slope seems to be negative because it's a decreasing line.

Interviewer: Why did you consider the tangent line in the y direction?

Hamid: Because the first derivative is respect to y.

Interviewer: Okay, continue.

Hamid: I go in the x direction because the second derivative is respect to x. If I draw the tangent line at the point (3,0) in the y direction it will be umm a line like this and it's a horizontal line. Comparing the value of the slopes between these two lines we see that their changes is positive so the answer is positive.

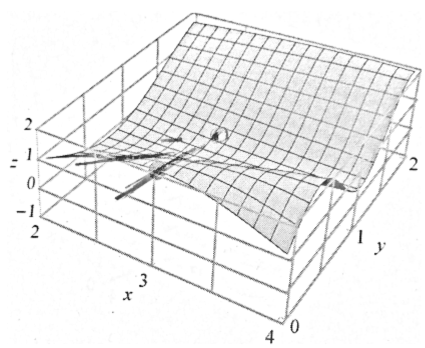


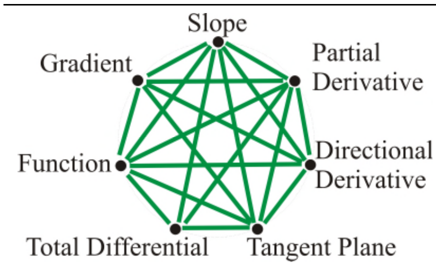
Fig. 3. Hamid's work on the question.

Hamid was given a function (F) and was asked about a (second) partial derivative (PD). He mediated his answer using slope (S) ("I need to compare the slope of tangent lines…"). So, this relation is labeled F-S-PD. His answer is normatively correct and justified, so that the relation is at least a transformation relation (F-S-PDt). Further (not shown here), his answers to questions involving F, S, and PD showed that he could move fluently between these components interpreting properties of one in terms of the other. This was done consistently and in a normative way throughout the interview. F-S-PD was labeled a conservation relation (F-S-PDe).

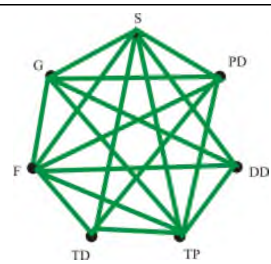
We then considered students' Schema development using the Schema development graphs in Table 1. Note that these graphs allow us to easily represent students' Schema development using the Intra-DC, Inter-DC, and Trans-DC triad. Results about students' Schema development can inform of their overall understanding and how to better support their learning. We don't show it in this paper but results about students' stages of Schema development may be used to compare different didactical approaches.

Table 1. Schema development graphs of the participating students.

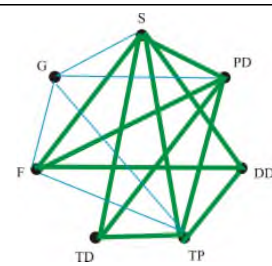
A1 Trans



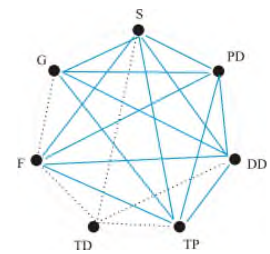
A3 Trans



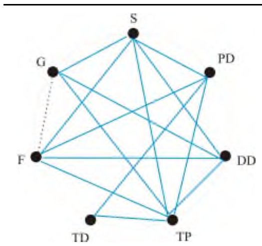
A2 transition from Inter to Trans



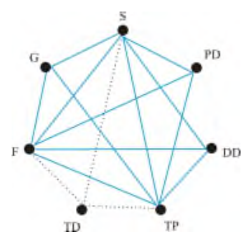
A5 Inter



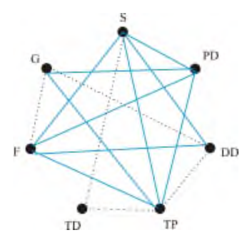
A6 Inter



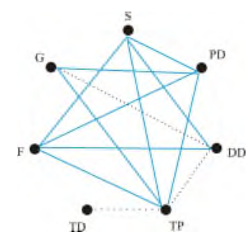
A7 Inter



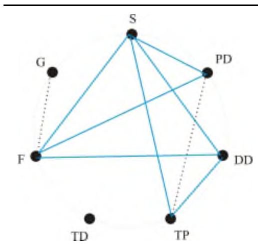
A4 Inter



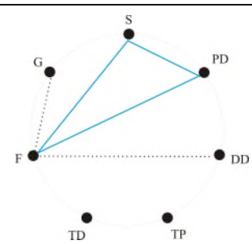
A8 Inter



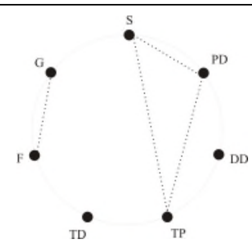
A10 transition from Intra to Inter



A11 Intra



A9 Intra



From Table 1, we can list the different relations between Schema components and the type of relation established by the students. This is shown in Table 2. This table will enable us to uncover the relations that seem to require more student support or increased classroom time or emphasis. This is important since it suggests possible problem situations to help students establish the relations. Thus, the investigation of students' stages of Schema development informs us about the overall performance of students and how to improve it.

Discussion and Conclusions

Our study contributes several observations pertinent to the didactics of the two-variable function differential calculus. Knowing what relations cause difficulty is useful. To see this, we consider the eight relations with the lowest total of conservation and transformation relations and consider the didactic implications of each one (Table 2). Each of F-G and DD-G have only four conservation or transformation relations. This suggests that more attention (class time and/or activities) has to be devoted to studying the properties of the gradient vector, relating it to the contour lines of a two-variable function or the level surfaces of a three-variable function, and to the direction of the largest instantaneous rate of change (largest directional derivative) of the function. There are only four TP-TD and two TD-F conservation or transformation relations. This suggests that the relation between a tangent plane at a point, the translation to the origin of such a plane and the total differential at the point has to be further explored, together with graphical comparisons of tangent plane and function. Students should be able to think of the total differential as approximating locally the change in functional values geometrically, symbolically, and numerically. There were only two PD-DD conservation or transformation relations. We conjecture that this results from how students constructed directional derivatives (see Borji et al., 2023). They did so by exploiting local linearity and using the tangent plane at a point to talk about slopes in the *x* and *y* direction rather than referring to partial derivatives at the point. This result suggests that it is important that students not only relate tangent plane and slopes for developing the directional derivative but also relate this to the usual formula for directional derivative as the dot product of gradient vector and unit direction vector. Students did not construct the relations TD-G and DD-TD. While they would be unusual constructions and thus unexpected for students, they provide ideas for activities to help students further the coherence of their Schemas.

Table 2. Relations and types of relations established by students.

Relation/type	Conservation	Transformation	Correspondence	No rel.
Slope-Partial Derivative	3	7	1	0
Slope-Directional Derivative	3	6	0	2
Slope-Tangent Plane	3	6	1	1
Slope-Total Differential	3	0	3	5
Slope-Function	3	7	0	1
Slope-Gradient	2	4	0	5
Partial Der.-Directional Der.	1	1	0	9
Partial Der.-Tangent Plane	3	5	2	1
Partial Der.-Total Differential	3	1	0	7
Partial Derivative-Function	3	7	0	1
Partial Derivative-Gradient	2	4	0	5
Direct. Der.-Tangent Plane	3	4	2	2
Direct. Der.-Total differential	0	0	1	10
Direct. Der.-Function	3	6	1	1
Direct. Der.-Gradient	2	2	2	5
Tan. Plane-Total Differential	3	1	4	3
Tangent Plane-Function	2	6	0	3
Tangent Plane-Gradient	2	6	0	3
Total Differential-Function	2	0	2	7
Total Differential-Gradient	0	0	0	11
Function-Gradient	2	2	6	1

Another contribution of the study is introducing the notion of the scope of an instrument. This allows examining the suitability of an instrument to explore Schema development. The study also contributes the notion of a student's Schema development graph. It gives a visual way to represent the Schema development of an individual at a moment in time. This simplifies the analysis of students' DC-Schema development using the Intra-DC, Inter-DC, and Trans-DC triad. It can also inform possible didactical interventions, as the above discussion shows.

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