Integrated Cognitive-Instrumental Model: A Dual Framework for Analyzing Cognitive Processes

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Influenced by Piagetian constructivism and Vygotsky's sociocultural theory, contemporary research has explored both individual cognitive development and the role of tools to mediate learning. In this theoretical report, we propose the Integrated Cognitive-Instrumental Model (ICIM), which synthesizes two prominent frameworks: APOS Theory (grounded in constructivism) and the Instrumental Approach (rooted in sociocultural theory). ICIM combines the strengths of both frameworks, where APOS Theory explains how learners move through stages in their cognitive development, and the Instrumental Approach shows how tools and artifacts shape and support this development. Together, these frameworks allow for a more comprehensive understanding of students' cognitive development by bridging the gap between Piagetian and Vygotskian principles and highlighting the interplay between individual cognitive processes and the socio-cultural environment. ICIM offers a robust, multi-theoretical approach that provides a new perspective into student thinking with the potential to inform both research and instructional practices.

Keywords: APOS Theory, Instrumental Approach, Constructivism, Sociocultural Theory

Introduction

Understanding student thinking has been the cornerstone of mathematics education research for over a century, dating back to the late 19th and early 20th century with Behaviorist theory focusing on observable behaviors and the external conditions under which learning occurred (Thorndike, 1922; Woodworth & Thorndike, 1901). While these early studies laid the groundwork for systematic educational research, they overlooked the internal cognitive processes that are fundamental to meaningful learning and understanding how to support such learning (Erlwanger, 1973). As the field evolved, the focus gradually shifted toward understanding the cognitive mechanisms that underpin students' learning processes, a shift that was groundbreaking in fostering a deeper comprehension of how students construct and internalize mathematical concepts. This shift in focus was significantly influenced by the work of theorists such as Jean Piaget and Lev Vygotsky. Piaget's writings, along with von Glasersfeld's interpretations (e.g., von Glasersfeld, 1984, 1987, 1995), introduced the constructivist approach, which emphasizes the active role of the learner in independently constructing knowledge/reality through a process of action and reflection. Vygotsky's sociocultural perspective, meanwhile, highlighted the critical role of social and cultural contexts in shaping cognitive development, particularly through language, tools, and social interactions with more knowledgeable others (Confrey, 1995; Pugh, 2017).

Drawing on these two theoretical perspectives, we present a framework with the goal of providing a straightforward mechanism to characterize students' in-the-moment cognitive activity. To achieve this, we draw on two frameworks that build on and connect Piaget's constructivist theory of learning and Vygotsky's sociocultural perspectives. In the following sections, we summarize the frameworks used, describe the merger of these frameworks into one integrated framework, and conclude this report with implications, considerations, and questions for the field. By integrating these perspectives, we hope to offer a more accessible and multi-theoretical understanding of student cognition in mathematics, with the potential to inform both research and instructional practices.

Frameworks

The first framework we are leveraging with this work is APOS (Action, Process, Object, Schema) Theory, grounded in constructivist principles focused on the mental actions, processes, and general cognitive structures involved in mathematical thinking. The second framework is the Instrumental Approach, informed by the sociocultural perspective, which emphasizes the role of tools and artifacts in shaping cognitive processes. In the following sections we present a review of the various components of APOS Theory and the Instrumental Approach, outlining their constructs so that they can later be synthesized together in an integrated framework that we are tentatively referring to as the *Integrated Cognitive-Instrumental Model* (ICIM).

APOS Theory

Drawing on Piagetian principles to characterize students' cognitive activity, we draw on the well-established cognitive framework known as APOS Theory (Arnon et al., 2014; Asiala et al., 1996; Dubinsky 1991; Dubinsky, & McDonald, 2001). APOS stands for Actions, Processes, Objects, and Schema, and describes a constructivist perspective built upon Piaget's theory of reflective abstraction. Central to the theory is the hypothesis that mathematical knowledge is the result of an individual's construction of four key components: a) mental actions, b) processes, c) objects, and d) schemas to make sense of mathematical problem situations. Each of the four key components of APOS Theory represent a distinct stage in the development of mathematical knowledge (Dubinsky & McDonald, 2001). Actions represent external transformations of mathematical concepts. These actions can be thought of as the application of explicit algorithms or memorized procedures. Importantly, actions lack justifications and are not typically connected to other forms of mathematical knowledge. When an action is carried out multiple times through repeated stimuli, and reflected upon, it becomes a process. Processes are perceived as internal, implying that they have meaningful connections to an individual's existing mathematical knowledge. This enables individuals to imagine a variety of transformations, omit steps in their conceptualization of the actions occurring, anticipate results, and provide a justification for their processes. An object is constructed from a process when individuals become aware of the various processes that describe a central idea. More specifically, the learner realizes that transformations can act on the central mathematical idea itself, therefore treating the idea as an object that can be acted upon. The literature refers to these objects as higher-level mental constructions that extend beyond the context in which they were originally formed – in more theoretical terms, they are extensions of Piaget's notion of reflective abstraction (Arnon et al., 2014; Dubinsky, 1991). Lastly, schemas represent a coherent collection of actions, processes, objects, and other schemas that relate to a specific mathematical idea or central topic. Schemas are characterized by their interconnectedness, providing the learner with an internal framework which helps them determine whether a given mathematical situation falls within their scope (Dubinsky & McDonald, 2001).

For the purposes of using APOS theory to understand student thinking, the application involves a cyclical process consisting of *theoretical conjectures, assessment design, and empirical data collection and analysis*. The purpose of this iterative cycle is to continually refine conjectures about the mental constructions of the learner, and is necessary for understanding how they might reason about various mathematical concepts central to the topic of investigation. The theoretical conjecture phase begins with the development of a genetic decomposition, which is a set of mental constructions that students might make to understand a mathematical concept. For example, students may first construct an action-based approach of logical statements by typing logical statements such as ‘True and False,’ ‘False and False,’ and ‘True and True’ into a text-based programming software such as Python or R to investigate the outputs. This action stage may then be followed by a process stage in which the learner identifies the central idea that the ‘and’ proposition requires two true propositions to produce a True output. Once the learner identifies this central idea, they may then utilize this as a piece of a larger mathematical process, therefore treating the ‘and’ operator as a mathematical object. In a teaching experiment investigating undergraduate students’ conceptions of set theory and logic, students exemplified these action-process-object stages using the ‘and’ operator in Python as an object to generate the intersection of three sets (Martinez, 2022). This decomposition is based on the anticipated mathematical activity of the students, given the researcher’s understanding of the concept, and it is refined through the subsequent iterations of interactions with the learner. In studies focused on the teaching and learning of a specific concept, typically the goal is to guide the learner through the genetic decompositions proposed by the researchers (Arnon et al., 2014). It’s important to note that the progression through the various stages of APOS Theory is not linear. Some students, as they are engaging in the action stage, move to the process stage, but may move back to the action stage to investigate a particular case of the concept (Dubinsky & McDonald, 2001). Below I highlight another example of genetic decomposition for the concept of sets and subsets which we will return to in the next section.

Consider the following question: How can we determine the total number of subsets of a set, S, and is there a way to calculate this for any set? The genetic decomposition starts with the action stage; a student might approach this problem by explicitly listing all the subsets of some set, say $S = \{1, 2\}$. They might list the subsets as $\{\}$, $\{1\}$, $\{2\}$, and $\{1, 2\}$. At this stage, the student’s understanding is limited to the concrete actions of listing subsets, and they may not yet grasp the underlying combinatorial principles. As the student gains experience and reflects on the process, they might realize that there is a systematic way to generate subsets without listing them all individually. Ideally, they will begin to understand the process of binary choices: for each element in set S, they can choose to include it in a subset or exclude it. This stage involves interiorizing the process of subset formation. At the object stage, the student has a more abstract understanding in that they can perceive subsets as objects themselves and recognize that these subsets can be operated upon. They can imagine performing operations on subsets, such as finding the union or intersection of subsets. This stage involves encapsulating the process of generating subsets into a coherent concept of subsets as objects. At the schema stage, the student has a comprehensive framework for dealing with subsets. They have a collection of actions, processes, objects, and other schemas related to subsets that they can then use to determine when a problem situation involves subsets and use their schema to address various subset-related problems. As demonstrated, APOS theory can provide a structured approach to understanding how students develop mathematical understanding. However, APOS theory (and Piaget’s constructivist theory of learning in general) fails to address social factors such as students interacting with their environment (in the Vygotskian sense), as an integral aspect of one’s learning. More specifically, the use of tools and their influence on cognition is missing from APOS Theory. In the next section I describe the Instrumental Approach, a theoretical framework that artfully bridges the work by Piaget and Vygotsky to account for constructivist principles, and the use of tools to describe cognitive activity.

The Instrumental Approach

Grounded in Vygotsky's (1978) concept of mediated activity, the Instrumental Approach broadly serves as a framework for analyzing how individuals expand their cognitive abilities through the use of tools to manipulate their environment. While APOS Theory largely developed through mathematics education research in the United States, the Instrumental Approach emerged within the French tradition of mathematics education research (Artigue, 2002; Guin & Trouche, 1998; Rabardel, 1993a, 1993b; Vérillon & Rabardel, 1995; Trouche, 2004). Vérillon and Rabardel's (1995) work was particularly influential, as they distinguished between subject-object interactions and laid the foundation for understanding how instruments develop through the use of tools, or artifacts. As described by Roorda et al. (2016), artifacts can be a physical object such as graphing calculators, geometry measuring tools or other manipulatives, but they could also be graphs, computational programming environments, images, proofs, formulas, or any other objects that can be used to reason about a given mathematics task. As the learner incorporates the artifact into their activity to solve a mathematical task, the cognitive process of the learner interacting with the artifact transforms the process merely from a subject-object interaction (i.e. learners solving a mathematical task without the artifact) into what we call an *instrumented activity situation*, or an expanded cognitive function that would not otherwise be possible without the artifact.

As Martinez (2024) described, "the development of the instrument involves the learned ability to manipulate the artifact itself (proximal action), in order produce an extended, and meaningful, manipulation (distal action) of the environment in which these actions are taking place" (p. 5). The relationship between proximal and distal actions are central to the development of the instrument (Abrahamson & Bakker, 2016; Artigue, 2002; Trouche, 2004; Vérillon & Rabardel, 1995). Take the example of a child learning how to use a spoon to eat soup, the child needs to understand how to manipulate the spoon (proximal action), even as they keep an eye on the effect it has on the soup (distal action). On its own, an artifact is not an instrument. It is only through the interaction and use of the artifact to solve a mathematics problem does it become an instrument through the manipulation of the artifact to enact a meaningful change in the learner's environment. Figure 1 is a representation of the interactions that are taking place as a learner (subject) solves a mathematical task (object) using an artifact to mediate the cognitive activity which leads to the development of an instrument.

The process of transforming an artifact into an instrument is known as *instrumental genesis*, which involves a bi-directional interaction between the learner and the artifact (Artigue, 2002). What this means is that the learner may be afforded creative and interesting ways to think about a mathematics problem given the potential use cases of the artifact while at the same time being limited by the constraints of the artifact itself which shapes the ways in which the learner may go about solving the task. This two-way cognitive interaction is known as the *instrumented action scheme* (Trouche, 2004) and is rooted in Piagetian constructivist principles. As described by Vergnaud (2009) one's scheme is the "the invariant organization of activity for a certain class of situations" (p. 88). In other words, one's scheme describes the process by which learners apply their existing knowledge to familiar situations and how they adapt or reorganize their thinking when faced with new problem tasks.

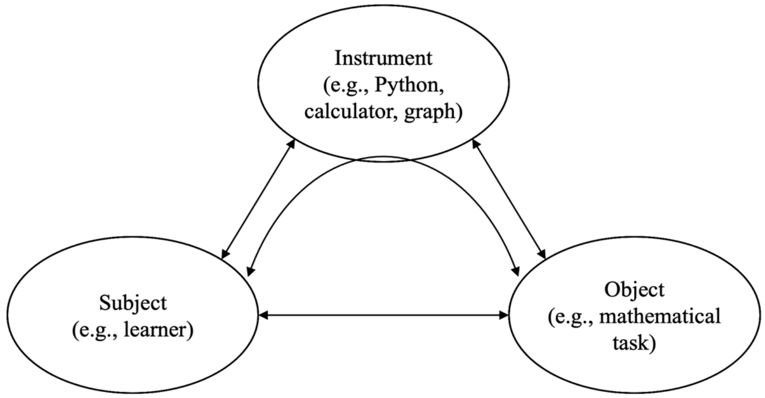

Figure 1. The subject, object, instrument interaction.

In the context of the instrumental approach, schemes consist of four main components (Trouche, 2004; Vergnaud, 2009), paraphrased from Buteau et al. (2020):

1. The goal of the activity: With sub-goals and expectations.
2. Rules of action: Stable behaviors of the learner.
3. Operational invariants: Which can be theorems-in-action (propositions considered as true) or concepts-in-action (mathematical concepts considered as relevant).
4. Possibilities of inferences: These possibilities are essential for the adaptation of the scheme to the specific features of the situation.

As Buteau et al. (2020) explain, a subject's mathematical activity can reveal these elements of their scheme when they state something like, "When I want to do this [goal], I always act like this [rule of action] because I think that [operational invariant]" (p. 1027). While students may not always articulate their reasoning so clearly, we can infer their thought processes through their observable behaviors. For instance, rules of action can be described as approaches the learner comfortably relies on to accomplish a familiar task, which are guided by their beliefs or understanding (operational invariants) of how things work. The operational invariants are divided into theorems-in-action and concepts-in-action. Theorems-in-action are the learner's constructed understandings of how things work, while concepts-in-action relate to the mathematical ideas important for the task. The possibilities of inferences refer to situations where learners modify their schemes by creating new rules of action or operational invariants.

Taken from Martinez (2023), let's consider the same problem scenario from the previous section in which a learner is asked to find all the possible subsets for any given finite set. Let's also imagine that the learner is using a computational programming environment like Python as the artifact which is developed into an instrument. In this case, one potential *rule of action* could be the creation of a loop that creates all the possible subsets. A *concept-in-action* could be the idea that a subset contains only elements from the original set and a *theorem-in-action* could be that sets are unordered, meaning $\{1, 2, 3\}$ is equivalent to $\{3, 2, 1\}$. As for a *possibility of inference*, encountering the empty set (which is not something that a learner using Python may naturally produce) would require the learner to create a new concept-in-action that includes the empty set as a subset.

It is important to note that these conceptual elements, which are inferred by the researcher, must ultimately be supported by the learner's observable actions. Naturally, some elements of a learner's scheme may be overlooked, but the goal is to identify the most relevant concepts tied to each rule of action. Together, these conceptual components form the learner's scheme, but as Roorda et al. (2016) describe, schemes can be observed at two levels of granularity. At the first level, a scheme may consist of a single rule of action and its related operational invariants, observable through one instance of behavior. At the second level, multiple schemes combine to form an instrumentation scheme related to a specific task (Roorda et al., 2016). In the case of calculating subsets, the learner's overall instrumentation scheme may involve several steps: creating an algorithm, storing subsets in a list, and determining the total number of subsets. Each of these steps can represent individual level-one schemes, while together they form a level-two scheme for solving the subsets problem.

Integrated Cognitive-Instrumental Model

We should note that neither of these frameworks makes claims about the permanence or stability of cognitive behaviors; however, they both provide fine-grained analyses of in-the-moment reasoning at the individual level. That said, from our perspective, each framework has specific weaknesses that could limit its utility in analyzing student thinking. In the case of APOS Theory, the notion of schema – intended to encompass all actions, processes, and objects related to a mathematical topic – can be overly vague. This can make it challenging for researchers, and especially practitioners, to extract practical insights about student thinking. Some researchers have worked around this issue by focusing more directly on epistemological nuances rooted in Piaget's notion of reflective abstraction, such as Sfard's (1991) concept of *reification* and Zandieh's (2000) idea of *process-object pairs*. However, since APOS Theory is grounded in Piagetian principles, we argue that its more pressing limitation is its lack of emphasis on how tools mediate learning, particularly when attempting to provide a holistic view of learners' cognitive processes. In recognizing this gap, we turn to leverage the Instrumental Approach, which accounts for the role that tools and artifacts play in shaping cognitive development. While the Instrumental Approach addresses how artifacts shape cognition, its focus on concepts-in-action and theorems-in-action can sometimes complicate the analysis process. Although valuable in theory, the emphasis on identifying every subtle cognitive detail can make it challenging to construct a coherent account of a learner's thought process as they engage in the instrumental genesis of the artifact.

By integrating both frameworks, we propose a theoretical stance that conceptualizes learning as the cognitive work learners do at the individual level as they move through the Action, Process, and Object stages of APOS theory, in collaboration with the work they do to instrumentalize artifacts to solve mathematical problems. To illustrate how these frameworks can work in tandem, we revisit the subset problem introduced in the previous sections to synthesize the key ideas from both frameworks and demonstrate their natural co-utilization. As a learner engages in the Action stage of APOS theory, they may write out the possible subsets of S – these are rules of action, or stable observable behaviors. The process stage may entail the identification of a combinatorial method of reasoning about the number of subsets a set has – this describes the development of a concept-in-action, for each element in set S, it can either be in a subset or not be in a subset. At this stage in particular, the learner may utilize computational algorithms (their developed instrument) to omit steps or provide additional justification for their ways of reasoning. Finally, as the learner internalizes these actions and processes into their cognitive understanding of subsets, they may encounter a new situation, such as the empty set, where they must determine whether this new situation aligns with their existing understanding, or requires the development of new rules of action and operational invariants. In Table 1 we provide a table that showcases the overlap of the two frameworks, supported by the theoretical perspectives of Piagetian constructivism as well as Vygotskian sociocultural theory.

Table 1. Integrated Cognitive-Instrumental Model.

Component of Learning	APOS Theory (Constructivism)	Instrumental Approach (Sociocultural)	Affordances of ICIM
Cognitive Development	Action-Process-Object stages describe individual cognitive development	Rules of action guide how learners engage with tools, instrumentalizing the artifact to manipulate their environment	ICIM integrates cognitive development with the mediation of tools
Concept Formation and Abstraction	Reflective abstraction leads to the development of schemas for understanding concepts	Operational invariants (concepts-in-action and theorems-in-action) are developed through tool use	ICIM integrates concept formation from both a cognitive and social/mediated perspective
Adapting to New Situations	Accommodation leads to the development of new actions	Possibilities of inferences are influenced by the instrumented action scheme	ICIM highlights how both internal and external factors influence a learner's ability to adapt

As Table 1 illustrates above, the ICIM addresses three main factors of learning: (a) cognitive development, (b) concept formation and abstraction, and (c) adapting to new situations. We argue that integrating the components of APOS Theory and the Instrumental approach across each of these three components of learning provides a more robust characterization of one's cognitive development while remaining flexible enough to account for two theories of learning.

Discussion

We see the ICIM to be particularly useful for researchers wanting to analyze student's cognitive thought processes related to computational thinking. This is particularly relevant given Lockwood and Mørken's (2021) recent call for undergraduate mathematics education research to investigate the role of machine-based computing in the teaching and learning of mathematics. The integration of the two frameworks provides a more holistic view, adding texture to how we can articulate a learner's cognitive thought process, and builds a unique bridge between Piaget's and Vygotsky's theoretical perspectives that do more than merely complement each other. As Simon (2009) described, "The availability of multiple explanatory theories and the use of multiple layers of analysis can, depending on the research, provide a richer set of constructs for accounting for observed phenomena" (p. 484). The ICIM draws on this idea, not only capturing the complexity of learners' individual cognitive thought processes, but also highlighting the crucial role of tools and social context in shaping these processes.

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