A Preliminary Design Space for Digital Tools

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The growth of educational technologies outpaces our understanding of how to promote students' mathematical development with such technologies. It has become increasingly important for researchers to problematize digital tool design so that educators seeking to enrich students' mathematical knowledge with the use of digital tools support, and not hinder, student learning. In this theoretical paper, we draw on existing research on digital tools for mathematics learning to propose a framework that situates to-be-designed tools within a "design space". We argue that an initial conceptualization of tools within this design space can help relate and distinguish tools from the literature that would be otherwise difficult to delineate or compare. We describe the spectra that comprise the digital tool design space, give examples of tools from the literature that fit within clusters of design space, and discuss the utility of such a framing.

Keywords: digital tools, digital task design, student thinking and learning

Over the past few decades, new digital technologies—combinations of hardware and software—have been developed to support mathematics teaching and learning. Over time, the nature of these digital technologies has shifted from standalone software packages designed to support students' learning about a particular mathematical topic (e.g., SimCalc) or provide a specific kind of interactive experience (e.g., Logo) toward interconnected webs of digital technologies utilized by teachers to accomplish a variety of functions. Teachers now use digital technologies for organization and implementation of the work of teaching (e.g., assigning work, producing worksheets, displaying student work), to connect with communities of other teachers that share materials and other resources, to individualize assignments and assessments of previously learned mathematics, and to support new ways of doing and representing mathematics (Sinclair & Robutti, 2020; Clark-Wilson et al., 2020).

At the same time, researchers have studied digital technologies in mathematics education from a wide range of theoretical perspectives and educational contexts. Given the fractured nature of the research literature on digital technologies for mathematics learning, it is important to develop theoretical tools that can be used to connect research findings across studies. This paper is part of a larger project broadly focused on the design of digital tools for mathematics learning at the undergraduate level. By *digital tools*, we mean instantiations of digital technology created within a software environment to support a particular learning goal. As part of our work, we recognized a need for a descriptive framework characterizing attributes of digital tools that could be used to identify commonalities between digital tools that were developed in different software contexts, from different theoretical perspectives, and for different pedagogical and mathematical goals. In this paper, we address the research question: *What dimensions of design can be used to characterize digital tools?* We provide a preliminary framework with three dimensions of design.

Research on Digital Tools for Mathematics Learning

Existing taxonomies distinguish between digital technologies for mathematics learning by classifying them according to their potential use in instructional situations (e.g., Dick & Hollebrands, 2011; Drijvers, 2015; Hoyles & Noss, 2009). For example, Dick and Hollebrands (2011) make the distinction between *conveyance technologies*, which allow students and teachers to present, communicate, and collaborate, and *mathematical action technologies* which "can perform mathematical tasks and/or respond to the user's actions in mathematically defined ways" (p. xii). Whereas this distinction concerns what the tools afford students and teachers, Drijvers' (2015) taxonomy classifies tools by the type of mathematical activity they support for students. Drijvers (2015) distinguishes between digital tools for *doing mathematics* or *learning mathematics*. In the category of learning mathematics, Drijvers (2015) further differentiates between tool use for *practicing skills* and for *developing concepts*. Lastly, Hoyles and Noss (2009) present four categories of digital tools that have differing potential to shape students' mathematical cognition: (1) dynamic and graphical tools, (2) tools that outsource processing power, (3) new representational infrastructures, and (4) tools with high-bandwidth connectivity.

These taxonomies can be useful when networking individual research studies focusing on students' learning with digital tools embedded within a specific software environment. For example, Bray and Tangney (2017) used an elaborated version of Hoyles and Noss' (2009) taxonomy to systematically review empirical research on the use of digital technology in mathematics education. "Such taxonomies prompt researchers to define more deeply the technology under scrutiny to support the reliability and generalizability of findings, a factor that is becoming increasingly important due to the growing complexity and interoperability of emerging technologies" (Clark-Wilson et al., 2020, p. 1126).

However, these taxonomies also tend to be used to describe particular software packages (e.g., GeoGebra), types of software environments (e.g., spreadsheets, clicker response systems), or types of hardware (e.g., interactive whiteboards; motion sensors). This makes it challenging to see how or when findings from one kind of digital technology can be usefully applied to the design or analysis of another. For example, Baccaglini-Frank and Mariotti (2010) studied students' conjecturing processes when engaging with open problems in a dynamic geometry environment. They proposed Maintaining Dragging as a mode of interaction with draggable points that has the potential for supporting particular cognitive processes in the context of Euclidean geometry. For researchers studying digital tools that are not embedded in dynamic geometry environments, there is still quite a bit to learn from this study.

To effectively gain such inter-environmental and inter-software insights, we need theoretical tools that can be used to identify the relevant attributes of digital tools in such a way that they are abstracted from the particularities of the software environment. We propose a framing of *design space* to accomplish this goal. We situate our initial conceptualization of design space using examples from the undergraduate mathematics curriculum because of the rich collection of mathematically complex tools constructible from the undergraduate curriculum. We use definite integration as a primary context to delineate regions of design space according to our identified spectra.

Theoretical Framing

Research on digital technology in mathematics education draws on various theoretical frames (Lagrange et al., 2003), with individual studies often poorly connected to prior work outside of the theoretical framework being utilized (Artigue et al., 2009). In an attempt to integrate disparate theoretical approaches to the study of digital technologies for mathematics education, the TELMA project endeavored to use specific and shareable language that is “not dependent on a particular theoretical frame, allowing us to communicate” (Artigue et al., 2009, p. 220) across studies examining related phenomena from different theoretical perspectives. To accomplish this, the TELMA group introduced the notion of *didactic functionality* (Cerulli et al., 2006) to describe digital tools for educational purposes. The construct of didactic functionality has three components: (1) the set of tool features/characteristics, (2) a specific educational goal, and (3) a set of instructional modalities for employing the tool in an instructional context toward the desired learning goal. The framework we propose in this paper addresses the first component of didactic functionality by providing a descriptive language for the characteristics of digital tools.

In this paper, we utilize Yerushalmy’s (2005) notion of *design space* to theorize dimensions of design for digital tools. A design space is a structured way of representing the space of possibilities for the design of a tool. We conceptualize the design space for digital tools as a 3D space in which digital tools can be placed on each of three axes representing the components of our framework.

A Preliminary Framework for Describing Digital Mathematics Tools

In this section, we introduce a preliminary framework for describing digital tools for mathematics learning. The framework has three main components: (1) the Interactive ↔ Inert spectrum, (2) the Static ↔ Dynamic spectrum, and (3) the Specialized ↔ Generalized spectrum. As is implied by the word spectrum, we intend that digital tools can be characterized as somewhere along a continuous scale described by each component. Using the dynamic label simply means that the tool in consideration is closer to the dynamic end of the spectrum than the static end. Some aspects of a so-called dynamic tool will be more dynamic than its other, more static, components. Yet by having at least some dynamic components, the digital tool lies farther along the dynamic end of the static ↔ dynamic spectrum than if it did not have those components. Likewise, having more dynamic components, or having the same components with more emphasis on its dynamic parts, would place the tool even further down the dynamic end of the spectrum. However, while we characterize each of these three components as a one-dimensional scale, the factors that place tools on this continuum are not well-defined in a mathematical sense that would lend these scales to a total ordering. That is to say, we do not intend for this framework, in its current form, to be used for pairwise comparison of all digital tools along each of the components. A fine-grained comparison such as "Tool A, with two sliders and a moving graph, is more dynamic than Tool B, with one slider and two moving graphs" is not the intention of using the term "spectrum."

Finally, we do not propose a hierarchy of digital tool characteristics within this framework. The quality of a digital tool is determined by its coherence with the pedagogical goals and students’ current mathematical and technical capabilities. Although digital tools that are dynamic and interactive may afford students novel opportunities to engage with mathematical ideas compared to the paper-and-pencil environment, they are not inherently better or more valuable than static and inert digital tools. This framework is purely descriptive, and for each of the three spectra, neither pole carries a more favorable connotation than the other.

Interactive ↔ Inert

A key affordance of digital technology is interactivity. Kaput (1992) describes the difference between *interactive* and *inert* media:

> We characterize a medium as “inert” if the only state-change resulting from a user’s input is the display of that input. Thus the key difference with notations instantiated in

> interactive media is the addition of something new to the result of a user's actions, something that the user must then respond to. In inert media, the user can only respond to what he or she has directly produced. Any external response to the input must be made by a third party, for example, a teacher, tutor, or peer (p. 526)

We extend Kaput's characterization from a binary categorization to a spectrum. Digital tools can be categorized roughly as more interactive or more inert depending on the overall breadth and depth of their reactiveness to a user's input. For example, a video player has the ability to react to user input regarding stopping and starting of playback at various points in the video feed. However, a traditional video player does not have the capability to capture mathematical input and change the contents of the video feed based on that input. Embedding digital polling, or allowing for swapping of video feeds based on user input, would increase the reactiveness of the digital tool and move it farther along the interactive side of the interactive ↔ inert spectrum.

Static ↔ Dynamic

A second affordance of digital technology is the ability to support dynamic media (Kaput, 1992). With dynamic media, objects can change states as a function of time, which means "time can become an information-carrying dimension" (Kaput, 1992, p. 525). However, in static media, the state of objects remains unchanged over time. Kaput (1992) further suggests that a dynamism range exists, where some notions of "dynamic" are more strongly dynamic than others according to "continuous transition." In our characterization of the Static ↔ Dynamic scale, we expand on the scale's dynamic range.

Our Static ↔ Dynamic scale consists of four classes, ordered from most static to most dynamic: Static, Discrete, Weakly Continuous, and Strongly Continuous. The latter three classes refine "dynamic." To illustrate these four characterizations, we use a recurring example of a digital tool whose purpose is to help students understand the relationship between the definite integral and the area under a curve (see Figure 1).

A *Static* digital tool in this motif is a digital image depicting a generic curve with some area shaded under it, paired with the algebraic representation of a definite integral. We note that a static tool can still be interactive, allowing the user to take some action and get a response from the digital tool, but differing states of the mathematical objects on screen do not have temporal variation. The example in Figure 1a shows an interactive static digital tool that can be used to explore the relationship between a definite integral and the area under a curve. Students can click to place values on the upper limit of integration and see both the symbolic and graphical representations immediately update.

To transform this static tool into a *Discretely Dynamic* tool, imagine that instead of the ability to to click to place values on the upper and lower limits of integration students can drag points to adjust the upper and lower limits of integration on the graph (see Figure 1b). However, the motion of the draggable point is constrained to be only integer values. So when a student wants to adjust the upper limit of integration to be a larger value, they drag the point to the right and see the symbolic and graphical representation update through a series of discrete—but temporally organized—states.

To make this tool *Weakly Continuously Dynamic*, we remove the constraints from the draggable points so that students can adjust the limits of integration by dragging the points smoothly along the x-axis.

Finally, to make this tool *Strongly Continuously Dynamic*, it is converted into an autonomous animation rather than allowing user manipulation. In dynamic geometry software such as

Desmos or GeoGebra, this equates to a slider equipped with a play button that will enable variables to run through a prespecified range of values.

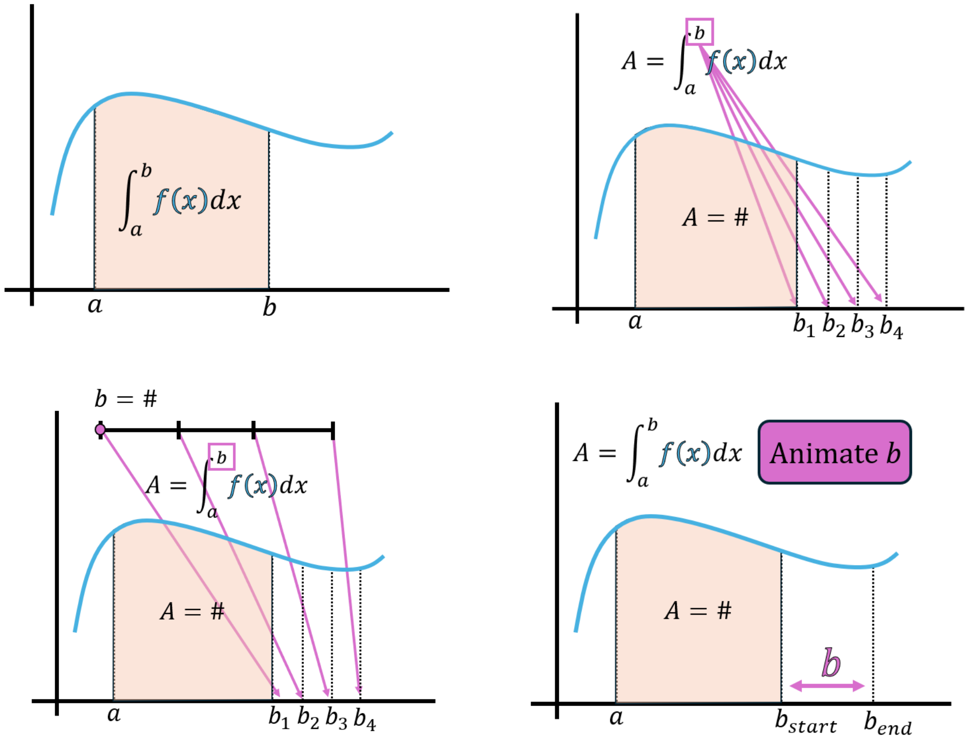

Figure 1: Variations on integration-based digital tools. From left to right: static, static interactive, discretely dynamic, strongly continuously dynamic.

These categories are not mutually exclusive. Each increasingly dynamic category is contained in the preceding static category. For example, a Discretely Dynamic tool has clear Static elements. Each state of the tool is itself a Static digital tool. Thus, when categorizing digital tools along the Static ↔ Dynamic spectrum, we consider the tool's *maximum potential dynamism*—its most dynamic classification within the constraints of its design or software platform.

Specialized ↔ Generalized

Cerulli et al. (2006) argue that digital tools are always developed in relation to some pedagogical goal. We characterize digital tools as *specialized* if they are developed in relation to a specific learning trajectory and *generalized* if they are developed for more general use across a range of pedagogical moments and/or mathematical topics. Examples of highly generalized digital tools are dynamic geometry environments, graphing calculators, and spreadsheet environments. These kinds of digital tools place relatively few constraints on the user's actions and are not necessarily designed to support students in learning a particular mathematical topic. A tool becomes more specialized by placing constraints on students' actions, designing feedback that orients the student toward a particular pedagogical goal, or pairing the tool with a particular mathematical task.

The spectrum of Specialized ↔ Generalized can be elucidated with the example of a definite integral and its connection to the area under the curve. On the specialized end of the spectrum, a digital tool could be hard-coded with a given accumulation function and specific numerical values for the endpoints of the interval over which the area is collected. The graph could further be linked to a specific context, such as accumulation of distance. Should a designer choose to allow the endpoints to be moved through the frontend of the application, or allow students to choose different accumulation functions, then the digital tool can be used in a broader number of tasks. Yet further down the spectrum, if the accumulation of distance context is dropped, and all possible endpoints and functions are able to be edited by the user, then this can become a highly generalized tool, able to be used in almost any application of definite integrals. However, designing such a tool involves a tradeoff, as it is no longer as directly aligned with particular pedagogical goals for specific tasks.

Discussion and Conclusion

In this theoretical paper, we propose a framework that structures a design space for digital tools to support mathematics learning. Together, all three spectra create a three-dimensional design space in which digital tools can be characterized. To give an idea of how these dimensions coalesce, we give a few prototypical examples of digital tools, and how they would be placed in the framework in a two-dimensional subspace, shown in Figure 2. The Specialized ↔ Generalized axis has been flattened for readability. In the rest of this section, we discuss the implications of the framework and provide an illustrative example of how the framework can support generalization about the design of digital tools across studies from different contexts.

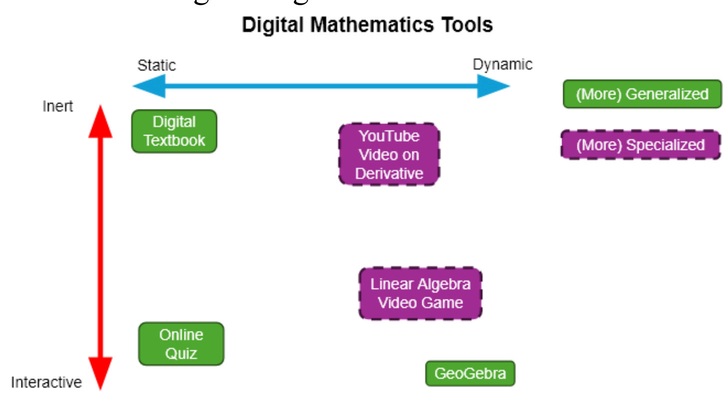

Figure 2. A two-dimensional representation of the design space structured by our proposed framework with examples of digital tool types.

One strength of this framework is its ability to distinguish between digital tools that are quite similar (e.g., built within the same software environment, developed to support the same mathematical goals). The examples shown in Figure 1 are all interactive digital tools that vary in their level of dynamism. According to existing taxonomies of digital tools, these tools would be quite difficult to distinguish. For example, each of the tools could be used to support students in developing concepts (Drijvers, 2015); they are all dynamic and graphical tools (Noss & Hoyles, 2009); and they are all built within a mathematical action software environment (Dick & Hollebrands, 2011).

Despite these similarities, each tool positions students differently with respect to the mathematical ideas at stake. The relationship between a student's perception of time passing and the changing state of mathematical objects embedded within a digital tool is under-theorized but beginning to receive more attention (see Moore, 2024). Early research indicates that the design of digital tools can be linked to students' conception of time as it relates to their conception of a situation involving covarying quantities (Margolis, 2024).

On the other hand, this framework can also be used to draw out similarities between digital tools that are otherwise quite different. Recall that Baccaglini-Frank and Mariotti (2010) wrote about students' interaction with draggable points used to define an interactive geometric diagram embedded within a dynamic geometry environment. Students interacted with the digital tool via a modality called Maintaining Dragging, in which they attended to the trace of the point as they dragged it in such a way that some geometric relationship within the diagram remained invariant.

According to our proposed framework, the digital tools in Baccaglini-Frank and Mariotti's (2010) study would be Specialized, Interactive, and Weakly Continuously Dynamic. In Figure 3, we show a digital tool which was designed to provide students with opportunities to engage in covariational reasoning with graphs and mental constructions of situational quantities (for more details about this task sequence, see Paoletti, 2019). Students are prompted to drag the coordinate point–intended to simultaneously represent the temperature and amount of water flowing out of the faucet–in such a way that only one knob moves at a time. To complete this task, students have to engage in the modality of Maintaining Dragging. And yet, the faucet tool is not built within a dynamic geometry environment and is not designed to support students in making conjectures about Euclidean geometry.

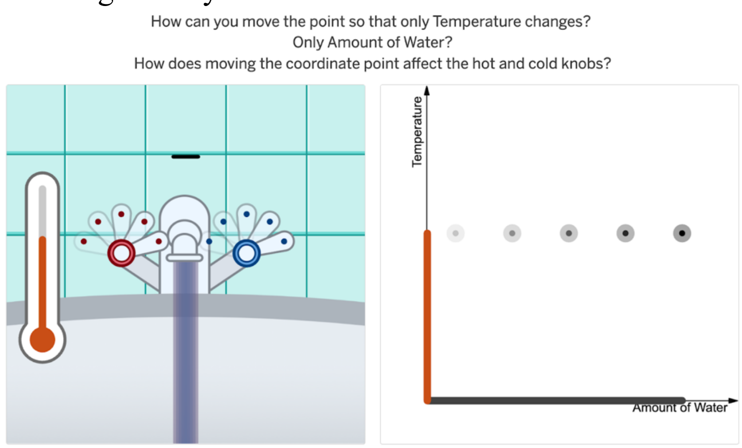

Figure 3. A static image of a student engaging in Maintaining Dragging (i.e., dragging the coordinate point in such a way that the temperature stays constant) with a digital tool.

Although we find this preliminary framework to be promising in our efforts to study the design of digital tools, we do not consider it to be complete. We anticipate that the framework will grow over time as we expand our theorization and analysis of the design of digital tools. For example, we anticipate that a category about the connectivity that digital tools afford communities of learners will be added to the framework soon.

Finally, we consider this framework to be useful for analysis of pre-existing digital tools while also being useful for the process of digital task design. We recommend further research on the affordances of tools across various locations in the design space for students learning.

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