Examining Ways in Which Students Engage With Sets of Outcomes in Combinatorics

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Lockwood (2014) articulated the importance of sets of outcomes, suggesting that a "set-oriented perspective" is a productive way of thinking for students as they solve counting problems. While this construct has indeed proven useful, it remains quite broad. In this report we aim to provide more detailed insight into the variety of ways students engage with outcomes while counting, exploring ways that we might elicit robust set-oriented perspectives in students. We present data from student interviews that were designed to elicit engagement with sets of outcomes, and we report on a variety of ways in which students worked with outcomes as they solved problems. We highlight contrasting cases that emerged in these interviews, making the case that there is room for nuance (and differences) in students' engagement with sets of outcomes in counting.

Keywords: Sets of outcomes, Combinatorics, Discrete mathematics

In combinatorics education, researchers have drawn upon the notion of a set-oriented perspective, which is "a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (Lockwood, 2014, p. 31). This perspective has been used in a variety of combinatorial contexts, including combinatorial proof (e.g., Lockwood, Reed, & Erickson, 2021) and computing (e.g., Lockwood & De Chenne, 2020), and it has been highlighted as a potentially productive way of thinking that may help orient students toward successful counting (e.g., Lockwood & Gibson, 2016; Soto et al., 2022).

In such work, the set-oriented perspective itself has remained rather broad, a high-level construct that promotes a general approach toward attention to outcomes. However, studies have (even implicitly) revealed variety in students' use of outcomes. Students may largely attend to specific outcomes (e.g., Lockwood & Reed, 2020), students may create partial lists of outcomes, or they may partition entire outcome sets (e.g., Lockwood, 2014). Based on such observations, we were motivated to better understand and explore ways in which students might, in practice, engage with sets of outcomes. Doing so would help paint a more nuanced picture of students' engagements with outcome sets, and might eventually enable us to elaborate and clarify the construct of a set-oriented perspective. We are broadly trying to answer the following research question: *In what ways do undergraduate students engage with sets of outcomes as they solve counting problems?* More specifically, in this report, we address the question: *In what different ways do students leverage sets of outcomes to solve counting problems?*

Literature Review and Theoretical Perspectives

Sets of Outcomes and a Set-Oriented Perspective

We frame our work within Lockwood's (2013) model of combinatorial thinking. When solving counting problems, students variously attend to *formulas/expressions, counting processes,* and *sets of outcomes*. Formulas/expressions are numbers, operations, variables, or relationships that yield the cardinality of some outcome set. Counting processes are real or imagined procedures for enumeration, and outcome sets are the elements being enumerated by a counting problem (Lockwood, 2013). In this paper we focus on the ways that students engage with the set of outcomes while solving various counting problems. Aligned with Lockwood's model is the notion of a set-oriented perspective. Harel (2008) described a *way of thinking* as a cognitive characteristic of a mental act; we take a set-oriented perspective (Lockwood, 2014) to be a way of thinking about the mental act of solving counting problems. This construct builds upon Lockwood's (2013) model, in which sets of outcomes are a central component. Lockwood and others have used this set-oriented perspective to frame students' combinatorial reasoning.

Prior work shows evidence of students reasoning about particular outcomes as they solve counting problems. Describing an equivalence way of thinking in combinatorics, Lockwood and Reed (2020) gave instances in which students compared and contrasted particular outcomes with the goal of determining whether two outcomes should be considered equivalent. When solving a problem about distributing identical lollipops to children, the students contrasted outcomes {1,2,3} and {3,2,1} (which children received a lollipop). They realized the two were equivalent, saying, "So you would be counting this [{1,2,3}] in a different combination, but really it wouldn't be an additional combination, because it's all the same, so this [{3,2,1}] is pretty much the exact same thing. It's just 123. […] because all the lollipops are the same" (p. 11). This is one way in which students may engage with outcomes and was integral to Lockwood and Reed's (2020) students' developing a set-oriented perspective.

In another example from Lockwood (2014), we see a contrasting way in which students reason about sets of outcomes. Here, a student was trying to count 8-letter passwords with at least 3 Es, reasoning that he could count all possible 8-letter passwords and subtract those with 0, 1, or 2 Es. We infer that the student partitioned a set of all possible passwords into subsets of desirable and undesirable passwords. In this common "total-minus-bad" approach, students engage with the set of outcomes by conceiving of the desired set in relation to some broader set, a different usage than the identification of particular outcomes in the previous example.

Other studies have made the case for the value of focusing on sets of outcomes and a set-oriented perspective (e.g., Erickson, 2022; Soto et al., 2022; Wasserman & Galarza, 2019). However, none of these studies specifically identified qualitative differences in how students reasoned about sets of outcomes. Our goal in this report is to detail some of the different ways in which students may engage with sets of outcomes when solving a variety of counting problems.

Local vs. Global Perspectives toward Sets of Outcomes

In framing our results, we also draw upon a distinction between local and global engagements with sets of outcomes (cf. Hamdan, 2006 and Cook et al., 2022). We characterize a local perspective as focusing on one or more particular outcomes, without necessarily taking into account the entire set of outcomes. It is local in that the focus is on particular elements rather than the broader structure. In contrast, we characterize a global perspective on a set of outcomes as a focus on the set as a whole, rather than emphasizing only specific elements.

In Lockwood and Reed (2020), the student's comparison of outcomes 123 and 321 as being equivalent suggests a local perspective on the set of outcomes; the students focused on two particular outcomes and drew a conclusion about what was being counted and which formula to use. In contrast, Lockwood's (2014) student's use of the total-minus-bad strategy suggests a global perspective on the set of outcomes, not appealing to particular elements of the set but rather to some overall structure of the entire set. This distinction between local and global perspectives toward outcomes is one way to characterize students' engagement with the set of outcomes; it is one dimension across which we analyzed our data.

Engaging with Outcomes with Numerical vs. Structural Areas of Focus

We also highlight a distinction reflected in students' engagement with outcomes that relates to the focus of the outcome-related activity. In working with outcome sets, one may emphasize the cardinality of the sets, emphasizing or exploring the numerical properties, relationships, or patterns in those cardinalities. For example, Lockwood et al., (2015) reported students creating lists of outcomes among several small cases and noting numerical patterns among them. This suggests what we could call a *numerical focus* as they engaged with sets of outcomes. In contrast, students may focus not on cardinalities and numerical patterns, but rather on features or structural aspects of the set of outcomes. Students in De Chenne and Lockwood (2019) noted structure in lists of outcomes to determine the *nth* outcome in a given list. Here, a focus was on structural regularity in the set, and not on any numerical patterns. This distinction represents another dimension across which we analyzed our data.

Methods

The data we present is from a series of task-based clinical interviews (Hunting, 1997) conducted with 9 students (3 graduate and 6 undergraduate) at a large public university in the United States. We present results from interviews with Kyle and Erin (pseudonyms) – undergraduate students who had taken or were taking discrete mathematics at the time of the interview. These were part of a broader study exploring student uses of equivalence in counting.

The students solved various counting problems on an iPad and described their work, and both the students and the screen were videorecorded. During and after their problem-solving activity, the interviewer asked hypothesis-confirming questions to enable post-interview inference into the students' understandings of formulas, concepts, and strategies. If the students were unsure of a solution, or were unable to proceed after some initial work on the problem, the interviewer often encouraged them to try smaller examples of the same problem. To provide opportunities for students to engage with sets of outcomes, the interviewer would at times suggest exploration of a pre-generated outcome set (of a smaller case). For instance, the interviewer might have provided a pre-generated set of arrangements of 1, 2, 3, and 4, and the student could use the GoodNotes app on an iPad to select and drag any outcome to a different location on the screen, creating their own organizations of the provided outcome set.

Data Analysis

To identify ways in which students engaged with sets of outcomes, the second author reviewed all of the interviews and flagged episodes in which students specifically leveraged outcomes or outcome sets in their problem-solving process. (An example of students *not* leveraging outcome sets would be a student identifying the wording of a problem with a particular solution type, making statements like "This is a combination problem.") This researcher articulated broad themes and categories for the flagged engagements, drawing upon the local/global and numerical/structural distinctions. The first author then also reviewed all of these episodes that featured sets of outcomes. Together, via discussion, the first and second authors chose representative episodes to highlight illustrative distinctions among students' engagement with sets of outcomes.

For this paper, because of space constraints, we limit ourselves to presenting just a handful of episodes to highlight ways in which students engaged with outcomes. Also, especially given the fact that students were at times encouraged to work with whole sets of outcomes, we are not making claims about their spontaneous use of outcome sets, nor are we reporting on frequencies in a meaningful way to suggest that some are much more common than others. Rather, we seek to make the point that, even within this interview context where they were at times encouraged to engage with outcomes, students' means of leveraging outcome sets were far from monolithic. This all highlights the point that there are multiple ways in which students may engage with sets of outcomes, and allows us to progress towards elaborating nuance about what might be entailed in the way of thinking that underlies a set-oriented perspective.

Results

In this section, we provide some key distinctions in students' engagement with sets of outcomes. Because of space limitations, the goal here is to provide contrasting, detailed examples of students engaging with sets of outcomes.

Distinction 1: Local vs. global engagement with outcomes in a single student

We begin by contrasting two cases with Kyle, focusing on his work on two problems.

Kyle's work on the Groups of Students problem. First, we see Kyle's work on the Groups of Students problem: *How many ways are there to split a class of 20 into four groups of 5?* Kyle initially considered a process of successively selecting five students, reflected in the expression C(20,5)*C(15,5)*C(10,5)*C(5,5). But he noted that this process would yield redundancies, saying "That feels like just kind of the obvious way to go about it, but I feel like doing it this way is where I'm going to miss out on the redundancies." He went on to specify, "If we chose the first five, and then the second five, third five, and the fourth five. Yeah, and but then, just reversing the order where it's fourth five, third five, second five and first five, then that's redundant in that sense where it's not unique for groups of five, but it would be included in this [expression]." We interpret here that Kyle articulates one specific outcome generated by the process of successively choosing students, namely, one which partitions the students into 1-5, 6-10, 11-15, and 16-20. But he realized that the process would yield the same partition in different orders.

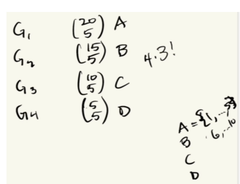

Figure 1. Kyle's work on the Groups of Students problem, in which he specifies a particular outcome

He continued working and specified the outcomes, assigning labels to the groups of A, B, C, D, as shown in Figure 1. He said, "Like if I said, uh, I would say A, B, C, D, then that's like, uh, one through five and then totally same for six and so on, so forth." Ultimately, Kyle realized he could arrange the groups of that partition in 4! ways and so the final solution would be C(20,5)*C(15,5)*C(10,5)*C(5,5)/4! (we omit details due to space). We claim Kyle's work here represents a local engagement with sets of outcomes. We infer this since he articulated specific outcomes - at no point attending to some broader organization of the entire set of groupings. Even more, he first explicitly articulated groups as a particular outcome – namely, splitting into four groups of 1-5, 6-10, 11-15, and 16-20, as groups A, B, C, D. So, he mediated attention to potential redundancies in the counting process by locally attending to specific outcomes and whether different arrangements were distinct.

Moreover, Kyle leveraged this local engagement to then generate the cardinality of the duplication, noting what counting processes might lead to similar arrangements of the outcomes. Notably, he did not just engage in numerical pattern matching, but considered what 3! and ultimately 4*3! might mean in terms of arranging specified groups of his specific outcome. With this local perspective, Kyle focused on forming and organizing iterations of a particular outcome, which he extended to the rest of the outcomes without explicit reference to a global structure.

Kyle's work on the Decreasing Numbers problem. In a contrasting example, Kyle was solving the Decreasing Numbers problem: *How many ways are there to form decreasing sequences of 3 numbers from the numbers 0-9?*. He did some initial work establishing cases based on the leading digit, but was overwhelmed by the number of cases. The interviewer then gave Kyle a set of outcomes to work with (as noted, this was an intentional intervention) – he was provided a written set of arrangements of the numbers 1-4 (Figure 2a), and encouraged to try and solve a smaller version of the problem: *How many ways are there to form decreasing sequences of 3 numbers from the numbers 1-4?* Kyle began by partitioning the set into four (ignoring the first digit), and highlighting the desirable decreasing 3-digit numbers, as seen in Figure 2b. This process led him to arrive at an answer of 4!/6, which yields the four desirable outcomes that he highlighted. In observing the set of outcomes, he noted, "Within each one of those permutations, only one of them is a decreasing sequence."

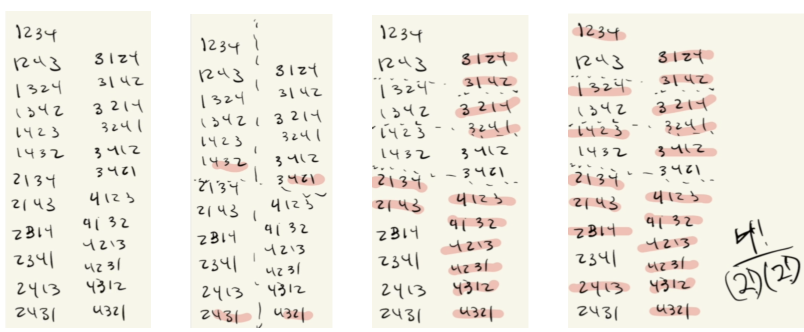

Figure 2a-d. Kyle's manipulation of sets of outcomes that were provided for him

Kyle expressed a desire to consider what might happen in other cases, so the interviewer prompted him to stay in the smaller 1-4 case but now consider decreasing sequences of two numbers. With a fresh list of outcomes, he again began to work with the set of outcomes. In Figure 2c, he partitioned the set according to the first two starting numbers, and he realized that he could exclude some outcomes. Here, he decided to highlight outcomes that were excluded, "this time I'm going to highlight the ones that I'm excluding." In Figure 2c, he first excluded outcomes that had the same two starting digits in different orders (so, for example he kept 1234 and 1243, and he excluded 2134 and 2143). Then, in Figure 2d we see within the remaining pairs that started with the same two digits, he excluded those whose last two digits were not in decreasing order (he excluded 1234 and kept 1243, for example). He related this process to his expression of 4!/(2!2!); the first division by 2! was for the arrangements of the first two digits, and the second division by 2! was for the second two digits. Kyle arrived at a correct expression having partitioned the set of outcomes and made note of equivalent outcomes in the set.

We feel that this episode shows a different approach to outcomes for Kyle, where he immediately started to partition the set into outcomes in which he was interested. Notably, though, he considered the desirable outcomes as they related to the entire set, and he then used equivalence and division to figure out a more general formula. We highlight this episode to show the same student reasoning about sets of outcomes in a qualitatively different way. Here, Kyle was adopting a global perspective – he started with the whole set and partitioned it, connecting the operation of division to his partitioning activity. We maintain that his treatment of the entire set of outcomes is different from his approach on the Groups of Students problem, where he built up the set of outcomes and considered particular elements.

Distinction 2: Different Global Perspectives Among Students

It may be natural to think that, if students had access to an entire list of outcomes, they would use them in similar ways. However, in our data, we found that even though we gave students entire sets of outcomes, they did not always use them in the same way. In contrast to Kyle's previous example, we look briefly at Kyle's work on the Towers problem and then Erin's work on the Passwords problem. In addition to noting the problem-by-problem variety of ways that students engaged with outcome sets, we also make the point that there was variety even in the context of the same intervention. Said differently, students did not use entire sets of outcomes for the same purposes.

Kyle's work on the Towers problem. The Towers problem states: *Silas is making "towers" from 3 green, 4 red, 2 yellow, and 8 orange blocks. Using all 17 blocks, how many different "towers" could Silas make?* Kyle became stuck initially, and the interviewer suggested he try a smaller case with 2 green and 2 red blocks. Kyle was provided the same list of arrangements of 1-4 as in Figure 2, but 1 and 2 were labeled red and 3 and 4 were labeled green. Seen in Figure 3a, Kyle first used the provided outcome set to pick out the 6 desired outcomes in the subcase, and then proceeded to hypothesize a formula/expression to generate the 6 outcomes (Figure 3b).

Figure 3a-c. Kyle's use of outcomes in the Towers Problem

In contrast to his work on the decreasing sequences problem, here the provided set of outcomes was used as a source from which the desired outcome set was generated. Then, the desired outcome set was used for its cardinality, and the first outcome "RRGG" was used as a representative from which Kyle conjectured a counting process (and the paired 2*C(3,1) expression) to generate the representative outcome. In this episode, we see Kyle using the outcome set to generate a known cardinality (6) from which counting processes and formulas/expressions could be hypothesized.

Erin's work on the Passwords problem. Similar to Kyle's work on the Towers problem, Erin engaged in numerically-motivated uses of the outcome set for the Passwords problem: *A password consists of four upper-case letters. How many such 4-letter passwords contain at least one E?* Erin initially approached this problem by first selecting a position for the E to go (C(4,1)) and then filling in any remaining position with 26 letters; thus her answer was C(4,1)*26^3. This overcounts, as passwords with more than one E are counted multiple times by this approach.

The interviewer again suggested a smaller case: *How many 3-letter passwords of A, B, C contain at least one A?* The interviewer provided an entire set of outcomes for this smaller case

(Figure 3a). Erin used this list as a means to count a total of 19 outcomes, and then spent an extended period of time attempting to find known counting formulas that calculated to 19, noting that 19 being prime provided a difficulty for her. After some time, the interviewer drew her attention to the case of AAA being generated three times in her original solution, and she solved the problem by generating the cardinality of each sub-case (passwords with just 1, 2, or 3 As), connecting the subsets of outcomes and the expressions that generated those subsets. She was encouraged that her listing activity and expression resulted in a count of 19.

Figure 3a-b. The set of outcomes provided to Erin, and her work on the problem

In this final case, Erin used sets of outcomes in multiple ways. First, she used the provided set of outcomes only for its cardinality, before attending to formulas/expressions exclusively. We note this not to criticize Erin's work, but to emphasize that simply giving students outcome sets is not sufficient for them to partition the set, manipulate it, seek some structural regularity, or even focus on particular elements within that set. Ultimately, Erin then produced her own full list of outcomes that reflected her imagined counting processes for the subcases. We see this as Erin taking personal ownership of the outcomes as relevant to her process of construction.

Discussion, Conclusion, and Avenues for Future Research

We have provided contrasting cases that illustrate different ways in which students engage with outcome sets. This highlights distinct approaches to outcome engagement that ultimately suggest a diversity of experiences students might have when developing a set-oriented perspective. We now discuss some potential implications of our findings.

One point of discussion is that much of the research has made a case for the importance of having students engage with sets of outcomes, but we have shown evidence that more nuance and distinction is needed to elaborate more specific ways in which students engage with outcomes and with sets of outcomes. We have shown some potential axes of distinction in engagement with outcomes, but more work is needed to operationalize distinctions between engagement types, particularly in pursuit of ways to elicit a set-oriented perspective.

We have also shown that simply providing students with opportunities to manipulate outcome sets is not sufficient to engender certain kinds of engagement with outcomes - having them partition, identify structural regularity, or even focus on certain outcomes did not always necessarily occur even when students were provided with organized lists of outcomes. More work can explore what kinds of supporting directions or activities might be effective in helping students to engage in productive outcome-related activities. Additionally, in providing outcome sets, we note that there are some limits to how many outcomes can be presented to students; this emphasizes the importance of leveraging smaller cases.

Further, we have in this paper mostly shown students' engagement with outcomes in just a few problems. There is more to be explored about how engagement with outcomes more broadly develops into a stable way of thinking and thus a set-oriented perspective (Lockwood, 2014).

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