Linear Algebra Student Thinking of Magnitude, Unit Size, and Coordinates

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This study explores how linear algebra students reason with three entities: magnitude, unit size, and coordinates in change of coordinate systems during teaching experiments. Two conceptual frameworks are employed in this study. The first framework, the coordinate system framework, was developed prior to the study and is used to frame a larger study that includes clinical interviews and teaching experiments. The second framework, the entities framework, was developed from the results of clinical interviews and is used to analyze student work from the teaching experiments. Students interacted with a GeoGebra applet that creates coordinate planes using matrices. The results present how students establish relationships among the three entities while working with the applet and how they represent their reasoning through matrix entries.

Keywords: Linear Algebra, Change of Coordinate System, Framework, Entity

Introduction and Literature

Coordinate systems are widely used in undergraduate mathematics. Students work with various coordinate systems, such as the Cartesian and Polar coordinate systems, and may switch between them in their courses. In linear algebra, students also encounter non-traditional linear coordinate systems that are similar to the Cartesian coordinate system but scaled and/or rotated. To graph a mathematical expression in a coordinate plane, students start by identifying coordinate pairs and comparing the components of each pair (a, b) with the unit length imposed by the coordinate system in each direction. (Lee, 2023) Observing this process, the author recognized the three entities that students engage with: coordinates, unit size, and magnitude.

Numerous studies have explored how students conceive of the magnitude of an object and how they interpret measures with changing units. (Steffe & Olive, 2010; Thompson et al., 2014; Vergnaud, 1988; Wildi, 1991). Attending to relationships of quantities or coordinating quantities has been a decent way to promote students' flexible reasoning with multiple coordinate systems in precalculus and calculus context (Joshua, Musgrave, Hatfield, & Thompson, 2015; Levenberg, 2015; Lee, 2017, 2020; Moore, 2015; Moore & Thompson, 2015). In linear algebra, change of coordinate systems is primarily discussed in the context of change of basis. Hillel (2000) pointed out that students tend to think of coordinate systems as the coordinate axes rather than something generated by a pair of basis vectors. Some studies concerned with students' difficulties in understanding matrix representations (Hillel, 2000; Hillel & Sierpinska,1994; Sierpinska et al., 1999). Specifically, Hillel & Sierpinska (1994) claimed that it is particularly hard for students to distinguish between the matrix representation of a linear transformation operator and the transformation itself. Wawro et al. (2013) developed a task sequence that helps students reinvent the [1]matrix diagonalization $A = PDP^{-1}$ in the context of Re-Naming points in a non-traditional coordinate system which uses y=x and y=-3x as the new axes.

Building on prior studies' insights into student conceptualizations of quantities, numbers, operations, and matrix representations, this report highlights the three entities (magnitude, unit size, and coordinates) and examines how students reason with them in coordinate planes. Given

[1] P is an invertible matrix and D is a diagonal matrix.

the increasing importance of linear algebra in STEM education (Tucker, 1993; Stewart et al., 2022), this study focuses on linear algebra students' thinking when working with non-traditional *linear* systems and matrix representations.

Conceptual Framing

Naming, Locating, Re-Naming, and Re-Locating

The coordinate system framework was developed from the author's calculus and linear algebra textbook analysis (Lee, 2023). I found that the types of examples illustrated in the textbooks handle coordinate systems differently, and that readers become engaged with different ways of thinking accordingly. The framework starts with two fundamental processes of representations: Naming and Locating. In Naming, a location in space is being measured following the convention imposed by a coordinate system and creates the measurement, a name. For example, a location in the 2D plane gets its name as (1,1) with the Cartesian coordinate system. On the other hand, in Locating, an existing name creates its location in space following the convention imposed by a coordinate system. An example of Locating is that the ordered pair (1,1) puts on a specific point in the Cartesian coordinate plane. Figure 1(left) illustrates that Naming and Locating are the reverse processes to each other. The two processes can be extended to represent an object with multiple coordinate systems: Re-Naming and Re-Locating. In Re-Naming, a location that has been measured by a coordinate system gets its new name measured in a new coordinate system that is laid atop the location. That is, the location previously paired with (1,1) is being renamed with $(\sqrt{2}, \frac{\pi}{4}) \approx (1.414, 0.785)$ in the same space using the Polar coordinate system (Figure 1, middle). On the other hand, in Re-Locating, an existing name creates two different locations in space depending on coordinate systems being used. The new location may appear different from the first, but they share the same name. For example, (1,1) corresponds to two locations: one defined by a horizontal and vertical distance of 1 each, and the other determined by a distance of 1 from the origin and an angle measure of 1 radian from the horizontal axis. (Figure1, right)

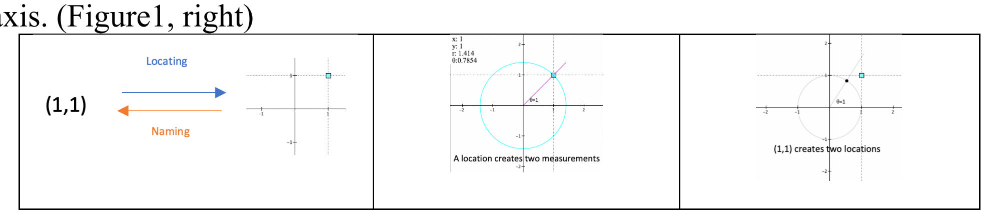

Figure 1. Naming and Locating (left), Re-Naming (middle), Re-Locating (right)

Three entities: Magnitude, Unit size, and Coordinates

The entities framework was developed from a part of a larger dissertation study (Lee, 2024). Through the process of Locating and Naming, I found that students engaged with three distinct entities in clinical interviews: magnitude, unit size, and coordinates. Magnitude refers to the size or extent of an object, characterized by its measurable attributes such as length, area, and volume. In Locating, students recognize the location of a coordinate as the distance of an object in each direction from the origin, which I specify as 'location magnitude'. Unit size refers to the measuring scale or dimension associated with each unit of measurement within a given coordinate system. When Locating, students compared a coordinate with a value 1 to determine the coordinate's location. In this case, unit size itself is a magnitude but is used as a measuring stick. Coordinates refer to number values assigned to the magnitude after measurement utilizing the corresponding unit size. The number value implies the amount of unit size. The three entities are conceptualized in a formula:

$$M(Magnitude) = U\ (Unit\ size) \cdot C\ (Coordinates).$$

The magnitude of an object is characterized by U (unit size), the measuring scale, and by C (coordinates), the amount measured. In Locating, the measured amount (C) is given, and then the location magnitude (M) is determined by multiplying the C by its measuring unit size (U). For example, in a clinical interview when locating an ordered pair (1.3, 0.5) on a plane, a student said "*the y* [0.5] *is the midpoint here* [points on the vertical axis]…*and*…*this is 1* [points to the unit in the horizontal axis], *then 1.3* [marks on the horizontal axis the right of 1]." The student needed the unit size of 1 to determine the magnitudes of 1.3 and 0.5 using multiplication either by scaling or through repeated addition. In Naming, the location magnitude (M) is fixed on the coordinate plane, and then its coordinate (C) is named by determining the amounts the measuring unit size (U) covers. For example, another student in the clinical interview rewrote the coordinate pair for a location $\begin{bmatrix}1.3\\2\end{bmatrix}$ as $1.3\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}$, breaking it down into components along the unit vectors consistent with the conceptualized formula. This formulaic idea extends to situations where the unit size changes to another size with multiple coordinate systems, leading to two scenarios emerged with switching from one coordinate system to another: 1. Under constant magnitude M: If U_2 (unit size for the second coordinate system) is k times U_1 (unit length for the first coordinate system), then C_2 (coordinate in the second coordinate system) will be $\frac{1}{k}$ times C_1 (coordinate in the first coordinate system). (Figure 2, left) 2. Under constant coordinates C: If U_2 (unit size for the second coordinate system) is k times U_1 (unit length for the first coordinate system), then M_2 (magnitude in the second coordinate system) will be k times M_1 (Magnitude in the first coordinate system). (Figure 2, right)

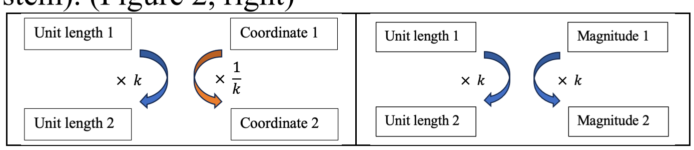

Figure 2. Re-Naming (left), Re-Locating (right)

This proposal reports on student reasoning with non-traditional linear coordinate systems in a teaching experiment to address the research question: How do students interact with various entities, such as magnitude, unit size, and coordinates when a coordinate system switches to another?

Methods

As part of a larger dissertation study, I conducted individual face-to-face teaching experiments with three students: Hann, Wilson, and Jeraldo. At the time of interviews, the participants were STEM majors who had completed Calculus 1 or 2 as prerequisites and had taken linear algebra at a large public university in the United States. Both their written work and interview conversations were recorded. The interview data were transcribed into spreadsheet and coded line by line, based on the author's entities framework. All the names used in Results are pseudonyms.

Description of the Applet (https://www.geogebra.org/classic/vz2mcxqr).

The teaching experiment utilized a GeoGebra applet created by the author. The applet was designed to engage students actively in creating a new coordinate system by modifying matrix entries. The applet features two panes: one on the left displaying the Cartesian plane in black, and one on the right showing the new coordinate plane in pink, generated by the matrix entries.

(Figure 3) Earlier part of the teaching experiments, the students interacted with the applet and learned how each entry in the matrix M (top left in Figure 3) influences the new coordinate plane on the right pane. They began to conceptualize the 2 × 2 matrix as a set of two column vectors, with each column vector representing a basis vector in the plane. For example, with $M = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$, the students recognized it as $\{\begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ generating a basis on the plane. (Figure 3, right pane)

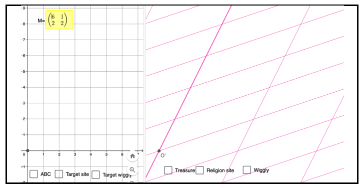

Figure 3. An example of the GeoGebra applet screen

Description of the Task

I designed the task protocols in three packets each called clinical interview, teaching experiment 1, and teaching experiment 2. This proposal includes Task 7 and Task 8 (Figure 4) in the teaching experiment 1 which aims to teach and learn matrix representations with a focus on individual entries. Previously, the students participated in the clinical interview tasks themed around Gulliver's Travels, where one of the main tasks was a treasure hunt, following the central idea of Realistic Mathematics Education (RME). During the clinical interviews, the students identified the location of the Treasure on Gulliver's coordinate system as (1.23, 2). In Tasks 1-6 of teaching experiment 1, the students engaged with matrix entry modifications using the GeoGebra applet and found that Gulliver's coordinate system can be generated using the two vectors $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus, the matrix M for Gulliver's gridlines is $M = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$. Task 7 (Figure 4) revisits the treasure and provides the students with a new coordinate pair (Josh's) for the same treasure, asking them to describe the new coordinate system. Task 8 then requires them to generate the coordinate gridlines using the 2 × 2 matrix. These tasks involve multi-step thinking commitments. Students determine what has changed (or not changed) in each component of the coordinate pairs, understand the implications of the change, identify the causes of the change, and modify the matrix entries accordingly to match the new coordinate pair.

Task 7: You just got a phone call from your friend, Josh, and he said that the treasure is at (1.23, 6) on his map, what does the Josh's map look like? (The treasure is at (1.23, 2) on the Gulliver map.)	Task 8: Find the entries of matrix M for Josh's map. In other words, the matrix M for Josh's map is $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$.

Figure 4. Task 7 and Task 8 (From Teaching Experiment 1)

Result

As shown in Figure 4, each task has its own goal aimed at examining student reasoning related to entities and matrix representation. In this section, we will explore the different ways that linear algebra student reasons with the three entities (magnitude, unit size, and coordinates) as they modify the matrix entries.

Hann: Reciprocal Relationship between Coordinates and Unit Size

Hann: I'm noticing that our x coordinates are the same, which means that our horizontal unit vector is the same. Um, so Josh's unit vectors… it's going to be the same as Gulliver, which means that our horizontal is 6 2 [$\begin{bmatrix}6\\2\end{bmatrix}$].

Interviewer: Why is it the same as Gulliver's horizontal?

Hann: Because we have the same x coordinate as Gulliver does. so that means I think it means it has to be the same… so now I just need to find his vertical vector and I'm noticing that it [coordinate]'s just *three times* what ours [Gulliver] is. So, for his [Josh] vertical [unit] vector, it's just going to be *one third* of ours, so it's going to be one third and two thirds, one third.

Interviewer: Why does it have to be reciprocal?

Hann: The location of the treasure [on Josh's map] is at six vertical units, whereas for us [Gulliver] it's two, so the only way that that can be true is that each of his units are three times smaller.

Hann recognized a reciprocal relationship between coordinates and unit size: enlarging the coordinate implies reducing the unit size. When the coordinate increases threefold from Gulliver to Josh, the unit size decreases to one third. His reasoning for multiplication relationship between unit size and coordinate is depicted in a diagram. (Figure 5) Then, Hann modified the second vector in the Gulliver matrix M by multiplying by 1/3.

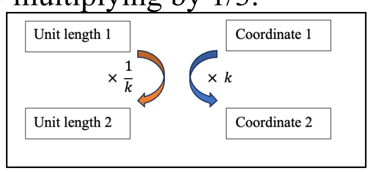

Figure 5. Hann: Reciprocity

Wilson: Proportional Thinking to Reciprocal Thinking with Coordinates and Unit Size

Wilson: His [Josh] vertical axis is three times what the value for I guess ours would be. This would probably look closer to 3 6 [$\begin{bmatrix}3\\6\end{bmatrix}$]… [after modifying the GeoGebra app] Oh, jeez.

I note that Wilson's visualization and verbal expression of Josh's vertical gridline was fine; However, when he tried representing the unit utilizing vectors, he modified from $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}3\\6\end{bmatrix}$ to the GeoGebra applet. Then, he encountered a discrepancy between his expectations and what he observed in the app. It appeared that Wilson reasoned that the unit needed to be scaled by a factor of 3, as the coordinate changed three times from 2 to 6. Consequently, he modified the second column from $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}3\\6\end{bmatrix}$, managing unit length and coordinate at a proportional manner (Figure 6, left).

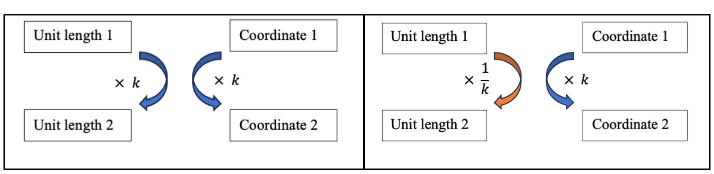

Figure 6. Wilson: Proportionality (initial reasoning, left) and Reciprocity (resolved later, right)

When Wilson discovered the discrepancy between his anticipated grid lines and the actual grid lines after modifying the entries of M on the applet, I followed up on him:

Interviewer: So, when you just hear the two coordinates, (1.23, 2) in Gulliver's while the treasure location is right here (1.23, 6) on [Josh] map, it has the same x value, right? The second coordinate is six now. When you're picturing the grid line using these two coordinates for the same location, what kind of image you have in terms of the grid line?

Wilson: There [Josh's map] would be an extra … lines… [then modified the second column in the app to 12 6 [$\begin{bmatrix}12\\6\end{bmatrix}$].

Interviewer: Why did you make it to 12 and 6?

Wilson: So, with the horizontal point being the 1.23 being the same between both maps, that's where my thoughts are thinking. That first column isn't going to change. That second column is where the big change will be. I was initially thinking with the 6 and 3 and 12, that's something along those lines could change, could scale it in a way that it is on the six…So, I think the point I need to find is what would one that would scale it down or closer….[later] What I think needs to happen for is that it needs to keep the same slope, … but also needs to be scaled down. So maybe about one third Oh, so maybe one third, two thirds because it [coordinate] needs to be scaled two, three, four, five, six.

In this episode, Wilson initially attempted to modify the entries of the second column using larger numbers, such as $\begin{bmatrix}3\\6\end{bmatrix}$ then $\begin{bmatrix}12\\6\end{bmatrix}$. It seemed that he continued to reason proportionally between coordinates and unit size when Re-Naming the treasure location. Upon discovering its invalidity of this approach with the applet, he experimented with small numbers while maintaining the same ratio to ensure that the slope of grid remains constant. Eventually, he figured out that the entries should be 1/3 and 2/3. He realized that to scale the coordinate up from two to six, he needs to scale the unit size down by one third, discovering the reciprocal relationship in matrix representation.

Jeraldo: Reciprocal Relationship between Coordinates and Unit Size

Jeraldo: On the horizontal direction, it stays the same. They have the same exact, um, the same exact number in the first entry [column]. The only difference I would say is that it probably looks the same with the angles and everything like that. It's just a little bit smushed down. So, there would be more lines… I'm thinking of smashing this down by a factor of three, but it has to be the same exact line [same slant]. Because if I change the entries [not holding the same ratio], then that would change the vector. So, if we keep one, two [$\begin{bmatrix}1\\2\end{bmatrix}$], it looks like this [same slant], right? But if I change this entry [not holding the ratio, one two], it would be at a different angle and completely different.

Jeraldo employed reciprocal reasoning between two entities, unit size and coordinate, in Task 7. Given the Treasure coordinate pair (1.23, 2) and (1.23, 6) in the Gulliver coordinate system and in the Josh coordinate system, respectively, where the vertical coordinates are tripled from 2 to 6, Jeraldo focused on the ratio of the vector components. He emphasized that the ratio needs to remain the same for the Josh's unit vector to hold the same slant of grid lines. He divided the Gulliver's unit vector $\begin{bmatrix}1\\2\end{bmatrix}$ by the scaling factor 3 to make it smooshed down. (Figure 7, left)

[On the second day of teaching experiment when revisiting the same task]

Jeraldo: So if I can keep this same line and smush it down, then we would able to have this [coordinate of 2] to be equal at six. So, I imagine if we just scale this down by three…scale down every entry by three. So essentially, take the original unit vector [Gulliver unit vector $\begin{bmatrix}1\\2\end{bmatrix}$] and divide it by the same factor.

Interviewer: Not multiplying?

Jeraldo: I guess you could multiply it by <u>the reciprocal of the scale factor</u>, but in my mind it makes more sense. If he wants to switch from, then you're gonna divide it. But I'm not sure how I will find the scale or the scale factor, but I think this might be the closest I'll get from

Interviewer: Why did you pick the number 3?

Jeraldo: Just because 2 and 3 are factors of 6. I just assume since we already have 2 right there, we can use 3. (Figure 7, middle)

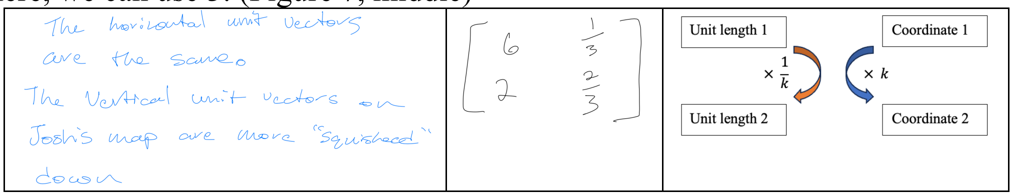

Figure 7. Jeraldo: Description for Josh's map (left), Matrix M entries (middle), Reciprocity (right)

Jeraldo recognized the reciprocal relationship: enlarging the coordinate implies reducing the unit length. His reasoning for multiplication between unit length and coordinate is depicted in a diagram (Figure 7, right).

Discussion

Reflecting on the entity framework, which initially began with the analysis of student work in clinical interviews, the formulaic idea, $M(Magnitude) = U\ (Unit\ size) \cdot C\ (Coordinates)$, extends to situations where the unit size changes to another size with multiple coordinate systems. This leads to two scenarios emerged with switching from one coordinate system to another (Figure 2).

First, under constant magnitude M: If U_2 (unit size for the second coordinate system) is k times as large as U_1 (unit length for the first coordinate system), then C_2 (coordinate in the second coordinate system) will be $\frac{1}{k}$ times C_1 (coordinate in the first coordinate system). Task 7 is designed to support students in thinking of coordinates in relation to unit size while keeping the location magnitude (the Treasure) constant across coordinate systems. According to my coordinate system framework (Lee, 2023), this task was developed under the Re-Naming perspective. As a coordinate system switches to another, the results show that students engaged with the three entities, establishing the reciprocal relationship between coordinates and unit size in Task 7.

Second, under constant coordinates C: If U_2 (unit size for the second coordinate system) is k times as large as U_1 (unit size for the first coordinate system), then M_2 (magnitude in the second coordinate system) will be k times M_1 (Magnitude in the first coordinate system). Even though Task 7 is better suited for the first scenario, where the location magnitude remains constant, students also demonstrated an understanding of the proportional relationship between unit size and magnitude. Specifically, when the unit size decreases to one third from Gulliver's unit to Josh's unit, the magnitude also decreases to one third (Figure 8), establishing a proportional relationship between unit size and magnitude: reducing the unit size leads to a reduction in magnitude.

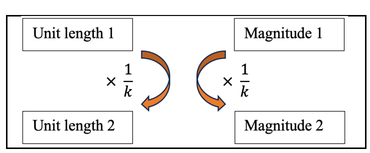

Figure 8. Proportionality between unit size and its magnitude

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