Multi-Team Embodied Activity and Semiotic Games to Develop and Tune Meanings for Abstract Algebra Concepts

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As students advance in undergraduate mathematics courses, the formal (e.g., textbook definitions and proofs) and informal (e.g., metaphorical) language they encounter becomes increasingly loaded. Additional work is needed to unpack how learners can engage in semiosis "meaning making" processes beyond surface marks, symbols, and verbiage. In this paper, we conduct semiotic bundle analyses to explore the potential of a full range of embodiments in meaning making. Our results show how embodiments can be leveraged in a more advanced abstract algebra setting to mediate semiotic conflicts between students' and institutional/communal signs and expand meaning-structures. We present more specific meaning tuning and elaboration strategies that utilized embodiments and were identified during our analyses.

Keywords: Embodied Instruction, Semiotic Bundle, Semiotic Games, Abstract Algebra

Introduction and Background

Much of the existing work on semiotics in abstract algebra settings is centered on (a) spoken language with puddle diagrams that instructors' produce and use to teach abstract algebra (e.g., Rupnow, 2021; Rupnow & Randazzo, 2024a) and (b) conflicts between students' signs and institutional meanings in separate interview settings (e.g., Melhuish, Ellis, et al. 2020; Melhuish, Lew, et al., 2020; Melhuish et al., 2023). Students can be proficient with formal language and symbols without being able to leverage this language to create new signs (Duval, 2017, 2006a; Sandoval & Possani, 2016). In an abstract algebra context, students can use metaphors for operation-preserving in ways that conflict institutional meanings (Rupnow & Randazzo, 2024b), experience conflict when translating a line in a proof to diagrams (Guajardo et al., 2021), and they can tear or erase parts of Cayley graphs when trying to create structure-preserving maps (Lajos, 2021). However, the interpretations that students attach to instructional language, and how they could be tuned towards appropriate interpretations in real-time, remains unclear. Melhuish et al. (2023) noted that "semiotic conflicts and resolutions are informed by contexts beyond the scope of our task-based interview study" and suggested that "additional research into the students' broader learning ecology" is needed (p. 515-516).

In research at the intersection of semiotics and embodiment in early algebra, "Gestures are highlighted as one of the ways in which mathematical cognition is embodied and they are elevated as a potentially important semiotic device in the teaching of mathematics" (Stylianou et al., 2024, p. 259). While attention to speech and inscriptions is prominent, the inclusion of a full range of embodiments (e.g., gesture, multi-body movements, etc.) and activities in which learners are free to collectively move in the classroom, is just beginning in abstract algebra settings (Gandell, 2024; Soto et al., 2024).

Such gaps along with findings on the semiotic affordances of gestures motivated this present work. We sought to devise and pilot an in-progress embodied instructional lesson for group theory topics (e.g., group properties, homomorphisms, quotient groups) to make students' interpretations for instructional language salient and to examine how students' and their instructor might leverage embodiments (e.g., gestures, multi-party enactments, verbiage,

diagrammatic inscriptions, materials such as necklaces, markers, etc.) to develop meanings and signs for abstract algebra concepts.

In this paper, we conduct *semiotic bundle analyses* (Arzarello & Robutti, 2010) on learners' evolving sign productions, that occurred during the embodied lesson. In addition, we explicate *semiotic games* (Arzarello et al., 2009) consisting of a sequence of intervention strategies that the instructor uses to tune and enrich students' meaning structures. Thus, we address the following two research questions.

1. In what ways do learners use embodied resources to construct meanings and signs related to algebraic concepts during multi-team embodied activity?
2. How does the instructor leverage embodiments in the classroom to tune and facilitate the enrichment of students' meanings and signs in real-time?

Embodiments Serving as Semiotic "Meaning-Making" Resources

We now turn to the literature to examine gestures (Alibali et al., 2019; Stylianou et al., 2024) and how other forms of embodiments (e.g., verbiage, inscriptions, whole body performances) (Nemirovsky & Ferrara, 2009) can be used by instructors and built into the classroom ecology to support meaning-making. Drawing from Vygotsky's (1978) work, Mariotti (2018) describes semiotic mediation as a "process" that "consists in the evolution from the emergence of personal meanings related to the accomplishment of a task and in the collective development/construction of shared signs related to both the artefact's use and the mathematics to be learnt." She highlights the instructor's role in integrating a variety of different tools into classroom lessons which engender different signs and personal meanings. One can also consider embodiments (Nemirovsky & Ferrara, 2009) as semiotic mediation tools (Alibali et al., 2019; Arzarello et al., 2009; Stylianou et al., 2024). From our review of the literature, we found that semiotic mediation via embodiments can be categorized into meaning-tuning and meaning-elaboration processes.

By *tuning processes*, we mean any *mediating* action that attempts to resolve a *semiotic-conflict* defined as, "any disparity or difference of interpretation between the meanings ascribed to an expression by two subjects (persons or institutions)" (Godino et al., 2007, p. 133) or any disparity among triadic signs for the same object (Duval, 2006). Alibali and Goldin-Meadow (1993) found that children who had evoked mismatches between spoken procedures and gestures, improved their performance after instructional interventions in which their incorrect interpretations were revoiced and pushed up against the meaning of the equal sign 'is the same as'. Thus, such mismatches should be viewed as a cue to the instructor that learners are at the "threshold of learning a concept" and that a learning opportunity is immediate (Alibali & Goldin-Meadow, 1993, p. 470).

By *semiotic elaboration*, we mean any mediating action in which a set of partial yet harmonious signs from subjects (persons or institutions) are preserved and pieced together rather than replaced. Alibali et al. (2019) describes cases in which instructors' regesture a learner's gesture and use speech to attach discipline specific language. In such cases, the instructor is using the learner's gesture as a semiotic resource to carry meaning associated with the gesture and attach it to disciplinary language. In addition, to the more prominent use of a gesture, speech, and inscriptions sets, multi-party formations/movements and material interactions can also serve as resources in the process of semiosis (Kelton et al., 2018; Kelton & Ma 2020). For instance, Kelton et al. (2018) found that families ambulatory movements of walking around a double torus together prompted verbiage from the children such as "Donut!" and a spatial sense for curves (p. 550). The natural embodied impulses of a child sliding, touching, feeling, and running freely without hesitation led to sense making moments that adult family members acting according to

proper normative rules (e.g., walking slowly and stopping to read exhibit signs) did not experience.

Theoretical Framework: Action Process Communication Space (APC)

The APC Space is a theoretical framework that brings together embodied cognition, communicative discourse, and semiotic meaning-making perspectives (Arzarello & Robutti, 2008). Drawing from an embodied perspective, this framework asserts that mathematical meanings stem from sensorimotor activity which is multimodal. Over time, metaphors which represent relational linkages between an abstract mathematical concept, such as adding, and physical activity, walking forward along a path, are created as new signs for already familiar concepts (Lakoff & Núñez, 2013). The communicative perspective emphasizes that becoming an "effective" communicator through mastery of written or speech "genres" (e.g., common patterns of communication) and their "meta-discursive rules" goes hand in hand with becoming an effective thinker (Sierpinska, 2005, p. 207-210). A semiotic perspective (Duval, 2017; Peirce 1931), emphasizes that meaning-making occurs in the process of creating a sign. Meaning is developed by and revealed through interpreting agents and their interactions as they modify previous signs to create new ones. The semiotic bundle is a "living" construct and analysis tool that integrates the three aforementioned approaches to mathematics education.

Semiotic Bundle and Semiotic Games

According to Peirce's triadic conception, a sign consists of the surface *representamen*, the *object* being signified, and the *interpretant* defined to be the underlying meaning structure attached to the *representamen-object*. *Representamens* are physical in that they can be observed through human senses. For any object, multiple signs can be produced, and each sign can be associated with multiple interpretants (Kehle & Lester, 2003, Figure 5.4, p. 103; Peirce 1931). *Semiotic system/register* refers to a set of signs, a set of rules for how to produce and transform signs, and relationships between signs and interpretive meanings (Arzarello & Robutti, 2010; Duval, 2017; Ernest, 2006). Signs can be transformed in at least two ways. A *treatment* refers to how a learner manipulates signs withing a single representational register, such as symbolic manipulations of an equation. A *conversion* is a mental mapping that converts *a sign* produced in an initial register to a *sign* for the same object in another target register. Additional *interpretants* may be afforded by converting across or *coordinating* various registers simultaneously (Duval, 2006; Sandoval & Possani, 2016). Meaning-making activities can also involve metacognition in which a previous interpretant, sign, object that has been consolidated, is then taken as the object of study for which a new sign and interpretant is produced (Kehle & Lester, 2003).

A *semiotic bundle* (Arzarello et al., 2009; Arzarello & Robutti, 2010) is defined as a complex adaptive structure that consists of all triadic signs produced by individual and collective bodies of meaning-makers, transformations of signs within and across production modalities (e.g., inscription, gestures, verbiage, etc.), and their relationships both temporal (e.g., co-occurrence of gesture and verbiage) and semantic. The semiotic bundle can provide information to the instructor regarding a peak learning opportunity and offers visibility into students' multimodal resources as they become available (Arzarello et al., 2009; Arzarello & Robutti, 2010; Vygotsky, 1978). In a semiotic game, students take a move by creating signs (e.g., representamen, object, and interpretant) and instructors take a move acting as mediators between student generated signs and institutional or classroom communal signs. Mariotti (2018) describes cases in which physical artefacts are used by an instructor as *semiotic mediation tools* to bridge semantic gaps between personal and communal signs. Similarly in thinking about the role of embodiment, we

seek to understand how students' embodiments performed during multi-team enactments can be used as a mediating tool in an abstract algebra classroom.

Participants and Settings

The researcher (second author) and teacher-researcher (first author) unpacked loaded instructional language to inform the design of an embodied unit consisting of four lesson days. We reflected on past teaching experiences, training in group theory, and the literature. We asked ourselves: what semiotic ingredients or resources could be leveraged to generate and tune meanings attached to metaphorical and formal language? We began to theorize possible viable paths that learners could take to develop personal meanings and signs. During Day 1 students were asked to enact a directed Cayley graph and were asked to consider how this representation encoded algebraic structures (e.g., closure, commutativity, connectivity). Tasks for Day 2 were designed to support students in embodying properties of elements within groups and embodying homomorphisms across groups. The students worked in three teams, associated with the groups D_4, $Z_4 \times Z_2$, and with mappings/homomorphisms. The eight students in each of the D_4 and $Z_4 \times Z_2$ groups were asked to develop an enactment for binary operations within these groups. Then the students were asked to consider how their embodied representation encoded algebraic structures (e.g., identity, inverses, associativity). Meanwhile, the mapping team was asked to develop a plan for testing if a candidate mapping was a homomorphism and lead the full-class congress in analyzing their potential homomorphisms.

Methodology

We audio-video recorded a total of four classroom periods that comprised the multi-team embodied unit as well as debriefing sessions. We prepared the audio-video classroom data that included agents, verbiage, gestures, inscriptions, glances (Arzarello et al., 2009; Sabena, 2008) collective body configurations and multi-party enactments (Nemirovsky & Ferrara, 2009; Kelton et al., 2018; Kelton & Ma, 2020) other socio-material interactions (e.g., with markers, Cayley table handouts, necklaces, and technology), whole body formations/movements and eye gaze (Soto et al., 2024). Agents included individual students moving as separate entities, pairs of students, and groups of students moving together as a single entity. We conducted two semiotic analyses, *synchronic* and *diachronic* (Arzarello et al., 2009).

We began the *synchronic analysis* by looking across modes of language production invoked simultaneously (e.g., verbiage, gesture, inscription, whole body movement, etc.). A column was added to code each time-stamp as either containing or not containing a semiotic conflict. Semiotic conflicts resulting from a mismatch in triadic sign components were highlighted by both the instructor and instructor-researcher. We discussed moments coded as conflict until we came to an agreement and provided a column in the coding book to take additional notes on the mediation strategies "games" and embodiments that the instructor invoked in anticipation or in response to semiotic conflicts. Following the synchronic analysis, we considered meaning-making across time and conducted a diachronic analysis.

During *the diachronic analysis* "the evolution of signs activated by the subjects in successive moments" is examined (Arzarello et al., 2009, p. 100). We kept track of any salient semiotic elaboration strategies that occurred across time such as representational *conversions*, *coordination* (Duval, 2006; 2017) defined as the linking-up of signs across time, and *embedding* (Kehle & Lester, 2003), in which the bundle at a previous time-stamp was taken as an object of study for which new *interpretants* and *representamens* are produced. We noted the time-span stamp and a description of the elaboration strategy towards enriched interpretations and signs.

Results: Day 2 Congress and Semiotic Games that Ensued

We focus our results to whole-class discussions during Day 2 [49:36.000-1:15:51.093] for this short paper. The instructor used the term "Congress" to refer to these discussion periods because these full-class discussions allowed the group to act like a legislative body that evaluates the evidence for claims and decides collectively if they are accepted. The instructor encouraged the sets of students to share their ideas and representations with a focus on the moments of friction in their work. To make progress on research question 1, we provide evidence of students' meaning-making through embodiments. To address part of research question 2, we present selected vignettes for "semiotic games" that clarify the instructor's tuning and elaboration interventions. For each vignette, we present findings on a) how the instructor engineered or responded to semiotic conflicts, b) embodiments that agents utilized as tools to tune or elaborate meanings, and c) the effects of such interventions on the living bundle.

Vignette 1 [49:36.000-54:06.520]: Engineered conflict between personal and institutional interpretants of homomorphism definition

This vignette showcases how the instructor leveraged the mapping team's inscriptions for injective mappings from D_4 to $Z_4 \times Z_2$ and embodied enactions to engineer and bring attention to a conflict between personal and institutional interpretations for a homomorphism. Namely that, it is necessary to check that a pair of elements satisfies the inscribed property equation, but it is not sufficient. The instructor opened congress with "Alright, I think we're ready to start acting. Are we ready to start commanding teams [wiggles fingers towards mapping team]. Cierra began brewing the engineered conflict as she led her classmates to physically act out the homomorphism property for a pair that would work. Mapping first and combining first led to the same embodied result, Taylor received the marker. Following this, Cierra exclaimed "It works and why is that?!" Seth grins and Cierra remarked, "That's because rotations are commutative!" The class collectively cheered "Yaaaaaaay!"

Next, Nico led the class to test the homomorphism property for a different pair, which the mapping team had planned and knew would fail the test. The choice of two rotations again is a counter to the "commutative" conflict. Nico began by pointing at the symbolic inscription of the homomorphism property on the board to direct classmates to **combine first then map**. He stated, "okay so the first group we're going to do R-ninety acting on R-ninety" and the instructor rephrased using the language "R-ninety with R-ninety". Four different signs emerged for this **combining first** computation. First, Max verbalized "double". Second, Sean gestured [makes fist with hands, half-circular motion, bumps left fist on top of right fist, and points to classmate]. Third, Nico gestured [points to domain bubble in puddle diagram inscription]. Fourth, Jez stacked two markers and handed the stack to Jacob. This fourth sign has the potential to carry the meaning that binary operations are mappings in themselves.

After combining signs were created, additional signs emerged for **mapping** R-180 to (1,0) which consisted of pointing gestures and maker passes. First, Cierra exclaimed "Jacob" [points to inscription (1,0) in range bubble]. Second, Nico followed with "and then R-180 is mapped to (0,1). Third, Jacob and Violet leveraged the previous two markers stacked sign invoked by X and chained it with a passing sign. Jacob, R-180, walked over to Violet [raises hands] playing the role of **(1,0)** and handed her the stacked markers which completed the combining first then mapping enaction. While nearing resolution of the engineered conflict a secondary conflict that the instructor had not anticipated occurred. The next vignette highlights the role of a collective body in mediating this unanticipated conflict.

Vignette 2 [54:06.520-55:16.993]: Utilizing an explosion of embodiments to tune Julia's unanticipated conflict

Nico continued the semiotic game and directed his classmates to **map first then combine.** Nico said, "Cool. Now, we're going to **map first**". Vivian pulled the stacked markers apart clearing the previous material sign. Nick leveraged board inscriptions and stated, "we mapped R-90 [point to puddle diagram inscription on board] to (0,1)." Next, Julia R-90 attempted to make a conversion from Noah's verbiage-inscription representation to a collective body representation. Julia walked towards Noah playing the role of (0,1). She extended her arm out and handed the marker to Noah. The instructor intervened with explicit verbiage, "Julia you need both of them" referring to the markers which were needed to make the more involved conversion from Nick's verbiage "we're going to map first". Julia paused her enactions with the marker and Noah. She then turned towards the instructor and asked, "Oh because I map to myself?", clipped the two markers together and looked back to the instructor. It is at this moment that the unanticipated conflict in the form of a verbiage-classroom sign mismatch had occurred. Julia verbiage "map to myself" and the classroom sign for combining (clipping markers together) were at odds. At this point it became apparent that the meanings Julia had attached to classroom signs and instructional verbiage needed further tuning.

Immediately following this conflict, a reactive explosion of arm movements and verbiage occurred spontaneously [54:26.860-54.31.440]. The instructor remarked "No, not together you're mapping both". At the same time Cierra said "you just keep holding them separately" [grips both fists, extends arms out in front of her torso, and holds fists in air] (Fig 1, left-right). Sean straightened his bent elbows and extended his arms in parallel as if holding two markers (Fig. 1, left). The effects of collective efforts to mediate Julia's mismatch follow. Julia responded "Oh" [holds both markers side by side]. Next, she passed the left marker to Noah then passed the right marker to Noah (Fig. 1 left-middle). It is at this moment that Julia re-attempted the conversion by replacing her enacted representation with the intended target representation. Following this, June extended her pointer and middle finger in parallel. This could be evidence that June converted the previous **mapping first** signs into her fingers [54:33.300-54.34.930] (Fig. 1, right).

Figure 1. Explosion of embodiments linked temporally to mediate Julia's mismatch

Noah then enacted the classroom sign for **combine** [stacked both markers together] and passed them to Erica-**(0,0)**. Cierra exclaimed "It's not the same person! That didn't work". The instructor remarked "Didn't work? Yes, curses". The instructor explained that "no bijection between them could be a homomorphism, because then they would be isomorphic so then both would be abelian or both would be non-abelian so it's good that you found a counterexample".

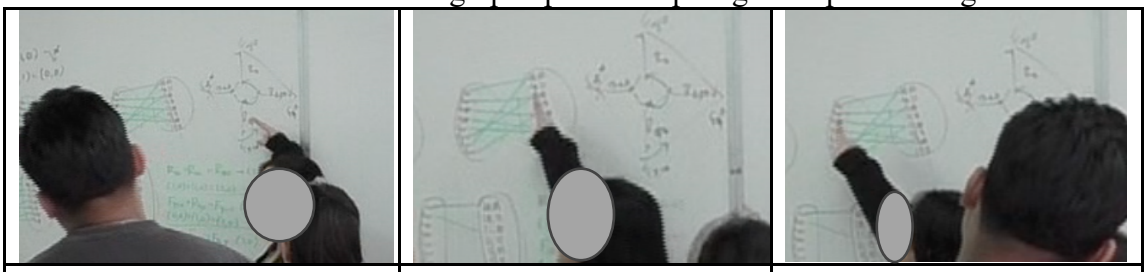
Vignette 3 [55:16.910-55:45.767]: Utilizing gesture to self-tune verbiage and convert signs

In this vignette, we first showcase how a student utilized gesture to self-tune her own verbiage. Cierra walked to the board and opened her body towards her classmates to share a more "promising" mapping. She explained, "This one we thought was a little bit more promising

[points to directed graph on the board] because we have the [pauses with gaze on board] generators make two inner circ uh [traces outer circle with right index finger into the inner circle]. Following her tracing gesture, Cierra automatically restarted her verbiage "uh an inner circle and outer circle [pinches inner circle with between thumb and index finger] with her gaze still fixed on the board. She spoke, "then some partnerships" [traces right index finger up and down over line segment connecting outer and inner circle of inscribed D4 Cayley graph]. This evidence suggests that Cierra's tracing and pinching gesture provided a mediating function. Namely, an awareness that her verbiage was incorrect. Her gesture was enough to help her self-correct and re-start her verbiage given that her gaze remained fixed on the board.

Cierra also created a compounded sign by coordinating the verbiage "partnerships", which emerged during Cayley graph enactions on Day 1, and the line segment tracing gesture over diagrammatic representations. Cierra then mapped meaning from her compounded "partnership" sign into the puddle diagram as she explained, "So we thought that the partnerships" [repeats tracing gesture] "would be mapping to the same" [traces finger across map line, points to (0,0) inscription in puddle range, and traces finger backwards] "coordinate on the other side, if that makes sense" [traces finger back to (0,0) in domain puddle] (Figure 2). Upon further enacted testing the class concluded that the "promising" mapping also failed.

Figure 2. Student coordinated directed graph "partnership" sign and puddle diagram



Repeats line-segment tracing for partnerships	Points at (0,0) in range of puddle diagram	Traces backwards to domain

Discussion

A primary contribution of this work is that it identifies the semiotic functions of an entire classroom ecology, in real-time during instruction. A secondary contribution is that this work attends to a broad scope of embodiments, in addition to those (e.g., speech and inscriptions) that are typically used to teach and conduct semiotic studies in abstract algebra settings. In this embodied lesson on homomorphisms, students and their instructor used a full range of embodiments to engage in meaning-making processes such as mediating conflicts between signs, making conversions between initial and target representations, and coordinating signs. Vignette 2 highlights how a collective body can mediate conflicts between a student and classroom sign. That is, both instructors and students together can successfully take on the role of a tuning mediator. In Vignette 2, the use of mapping leaders directing conversions between board inscription and multi-team enactions served as tool for drawing out an unknown semiotic conflict. In Vignette 3, Cierra created a compounded sign for "partnerships" which emerged during Day 1 to hypothesize and test a more "promising" mapping. Thus, a sequence of multi-team embodied activities can facilitate the compounding of signs and their respective meanings giving rise to a richer more meaning dense bundle. Thus, additional work that investigates the semiotic potential of multi-team embodied instructional units in advanced undergraduate settings is warranted.

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