Identifying Pedagogical Proof Moves

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Drawing on the concept of "moves" from English for Specific Purposes (ESP) genre theory, we identified pedagogical proof moves—moves proof authors make for pedagogical purposes—in a corpus of 40 abstract algebra proofs. We share 12 such pedagogical proof moves, presenting three in greater detail.

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There exists an image of mathematics texts as consisting of "pages of information with virtually no repetition, no varying of pace, few topic and concluding sentences, as few words as possible [… where] clarification is achieved by constructing very precise sentences without any extra words" (Tobias, 1989, p. 49). This observation about the terseness of mathematics texts could be made about proofs in particular—and *is* perhaps the image of prototypical proofs (see Dawkins and Weber, 2017, documenting the norm that "irrelevant statements are not presented in a proof," p. 131). Yet, in our experience, proofs, particularly those written for undergraduate students, are *not* solely presented as a chain of deductions without repetition or clarification. Consider the following example from Abstract Algebra lecture notes:

> **Theorem VI.2.** *Let (G,○) be a group and let a be an element of G. Then a has a unique inverse.*
>
> PROOF. Our proof follows the same pattern as the proof of Theorem VI.1, and you'll see this pattern again and again during your undergraduate career. Almost all uniqueness proofs follow the same pattern: suppose that there are two of the thing that we want to prove unique; show that these two must be equal; therefore it is unique. ((This paragraph has the phrase "*key trick!*" written in the margins next to it.))
>
> For our proof we suppose that *b* and *c* are both inverses of *a*. […] ((proof continues))

As we hope the reader will agree, the entire first paragraph is logically unnecessary. But the author's choice to include such "superfluous" elements indicates that they are not superfluous and do serve a purpose. Specifically, the paragraph's purpose appears to be to facilitate learning by distilling and pointing out the key idea behind most uniqueness proofs so that the learner may recognize the given proof as being an instance of this particular proof type.

To describe this first paragraph (and similar pedagogical elements of proof), we draw on a concept from English for Specific Purposes (ESP) genre theory: moves. Moves are functional units of a genre, that is, units which are focused on what a text is doing, not on what a text is saying or its form (Hyon, 2018). Moves have been studied since Swales's (1981) identification of a four-move structure in journal article introductions, later amended to a three-move structure (Swales, 1990) (e.g., establishing a niche). Relevant to our study is that moves: (a) may vary in size and do not need to align with grammatical units, (b) can be optional (i.e., not all moves need to appear in every text), and (c) can be flexibly used (i.e., moves need not occur in a specific sequence) (Bhatia, 1993; Hyon, 2018). With this in mind, we see the proof's first paragraph as realizing a move which one might call *commenting on proving*. Given our focus on the genre of proof and our interest in those moves that appear to serve distinctly pedagogical purposes (i.e., facilitate learning), we seek to address: *What pedagogical proof moves do authors of undergraduate texts make in their proofs?* To better describe these moves in our Results section, we will also draw on concepts from close cousins of ESP genre theory: (a) intertextuality from Bakhtinian genre theory, and (b) metafunctions from systemic functional linguistics (SFL) genre theory.

Relevant Literature

Mathematics education researchers have termed mathematical proof a genre (e.g., Lew & Mejía Ramos, 2020; Selden & Selden, 2014), a claim that we recently demonstrated aligns with modern genre theory traditions (Küchle & Dawkins, 2024). Since "[a] genre comprises a class of communicative events, the members of which share some set of communicative purposes" (Swales, 1990, p. 58), (slight) differences in communicative purpose can lead to (slightly) different genres. Therefore, we concur with Lai et al. (2012) and Lai and Weber (2014) that it is appropriate to view "proof[s] […] presented to students for pedagogical purposes" (Lai et al., 2012, p. 147) as a sub-genre of proofs.

This sub-genre itself contains some variation. Both Lai and Weber's (2014) and Pinto and Karsenty's (2020) mathematician participants expressed that features of their proofs for pedagogical purposes would depend on whether the proof was presented in lecture or written for a textbook. Both studies showed that mathematicians believed that textbook proofs should be rigorous and complete, whereas lecture proofs—due to lectures' affordances (e.g., reading the audience, gesturing) and constraints (e.g., time)—are allowed to omit details but should incorporate pictures, highlight main ideas, and communicate intuitions. Thus, pedagogical additions to proof (e.g., incorporating pictures, highlighting main ideas) appeared to be more associated with lecture proofs than textbook proofs for the mathematician participants.

Yet, Lai et al. (2012) demonstrated that in textbook proofs written for pedagogical purposes, mathematicians valued more than just rigor and completeness. By asking participants to improve the clarity of two proofs intended for a sophomore- or junior-level mathematics major, Lai et al. (2012) identified three types of revisions the mathematicians made: additions (e.g., adding a picture), alterations (e.g., altering phrasing), and deletions (e.g., deleting irrelevant statements). Further, Lai et al. (2012) confirmed mathematicians' beliefs that the quality of a proof presented for pedagogical purposes could be improved by: (a) "adding introductory and concluding sentences that highlight the proof's framework" (p. 158), (b) "highlighting the main ideas of a proof with the judicious use of formatting" (p. 158), and (c) avoiding the addition of "unnecessary or irrelevant assumptions or computations" (p. 158).

In sum, there is evidence that proofs for pedagogical purposes is a sub-genre of proof and that mathematicians are aware of it. Moreover, mathematicians appear to distinguish between proofs for pedagogical purposes in lectures and those in textbooks. In particular, mathematicians appear to find that several pedagogical ideas (e.g., incorporating pictures, highlighting main ideas) seem to have less of a place in textbooks than in lectures. At the same time, Lai et al. (2012) identified some additions, alterations, and deletions that mathematicians make to textbook proofs to increase their pedagogical merit.

Methods

Data Collection

To build a corpus of proofs, we selected ten random proofs from four Abstract Algebra texts for a total of 40 proofs. The four Abstract Algebra texts were selected to be similar in content but different in style so that pedagogical moves might stand out more. They were: (a) Abstract Algebra lecture notes that the first author remembered for their pedagogical merit (e.g., see the example at the start of the paper); (b) Gallian's (2017) *Contemporary abstract algebra*, which "[m]ay be the most widely used undergraduate algebra text" (Isaacs et al., 2015); (c) Dummit and Foote's (2004) *Abstract algebra*, chosen for its terseness; and (d) Carter's (2009) *Visual group theory*, chosen for its inclusion of visuals—common in mathematics, broadly, but, in our experience, less common in abstract algebra.

The 40 proofs were randomly selected as follows. For each text, a random number generator was used to select random pages. If a page contained no proof, we moved to the next randomly generated page number. If a page contained one proof, we selected this proof. If a page contained multiple proofs, a random number generator was used to select a proof from this page.

Data Analysis

As Lai et al. (2012) found, mathematicians add, alter, and delete texts (and pictures) from proofs to make them more pedagogical. Given our data (i.e., fixed proof texts), we decided to focus entirely on additively-realized pedagogical proof moves, that is, pedagogical proof moves that we were able to discern because they are realized through the addition of text. (Thus, the moves we eventually identified are a strict subset of all possible pedagogical proof moves.)

To identify these (additively-realized) pedagogical proof moves, we began by using a heuristic, by which we mean an "economical shortcut procedure[] that may not lead to optimal or correct results, but will generally produce outcomes that are in some sense satisfactory or 'good enough'" (Chow, 2015, pp. 977–978). In particular, we began by reading through the 40 proofs and crossing out as much text as possible to still leave a proof we judged to be logically valid. There were several instances in which we were unsure whether text could be crossed out, in which case we erred on the side of crossing it out. (Since we would later study this crossed out text more carefully, this means we erred on the side of overcoding.)

Studying the crossed-out text and asking questions around which text could be omitted to still leave a valid proof brought to our attention two types of moves, which we had felt unsure about crossing out from proofs:

- delimiting (Konior, 1993) moves (i.e., moves which indicate the beginning or ending of [a part of] the proof) (e.g., "Conversely, […]"); and
- justifying-supporting moves (i.e., moves which do not justify, but which provide justifying support by acknowledging that a warrant is needed) (e.g., "As in the proof of Cayley's Theorem, it is easy to verify that […] ((author omits argument))").

With awareness for these moves gained, we next coded each proof from beginning to end for:

- proving moves (i.e., moves which support the logical structure of the argument) (e.g., deducing, instantiating, defining);
- delimiting moves;
- justifying-supporting moves; and
- other (moves).

Given that moves can occur at various grain sizes (Hyon, 2018), our moves range in size from a word (e.g., "Conversely" realizing a delimiting move) to multiple sentences (e.g., the first paragraph of the Abstract Algebra proof at the start of this paper). Further, since text segments can perform multiple functions simultaneously, we allowed for the possibility of text segments realizing multiple moves at once.

We then focused our attention on the "other (moves)" code (to discern pedagogical proof moves) and coded its instances for their function(s) via process coding (Saldaña, 2009), whereby we described their function(s) using a verb or verb phrase (e.g., *simplifying notation*). We are currently engaging in pattern coding (Saldaña, 2009), that is, we are in the process of combining and consolidating our process codes into a smaller number of pedagogical proof move codes.

Results

Below, we present the three most common pedagogical proof moves in our corpus. We then briefly list the other identified pedagogical proof moves.

Reminding the Reader

The pedagogical proof move that occurred most often (30 times) and in all sources was *reminding the reader*. To further describe this move, we found the concept of intertextuality useful, which refers to "how texts draw upon, incorporate, recontextualize and dialogue with other texts" (Fairclough, 2003, p. 17). We observed authors reminding the reader by: (a) expanding on intertextual references, and (b) making eliminable intertextual references.

In the case of expanding on intertextual references, authors reminded readers of the details of the reference they were making (e.g., reminding the reader of a definition, process, notation). For example: After "Let's recall what the quotient process does to arrows between cosets," the author proceeded to describe the quotient process. This type of reminder saves the reader the potential work of looking up the details of the reference being made.

In the case of making eliminable intertextual references, authors drew a connection for readers to past work (e.g., "This is similar to the proof of Proposition 25"). Thus far, we have discerned two distinct pedagogical uses. First, authors pointed to past work to explicitly or implicitly leave part of a proof to the reader. For example: "This has almost the same proof as Fermat's Little Theorem. I'll leave the necessary modifications as an easy exercise." This way, the author provides the reader with an idea of how the statement could be proved. Second, authors appeared to make eliminable intertextual references to get readers to see similarities between proofs, possibly so that readers might draw on previous experience and/or objectify a proof approach (e.g., the start of the proof presented at the beginning of this paper).

Indicating the Difficulty of an Argument

In all four sources, authors used language that indicated the difficulty of an argument they were making. Yet, there appeared to be different purposes. First, we observed authors using words such as "clearly," "obviously," and "it is immediate" to serve as placeholder warrants (or warrants of self-evidence) (12 times). Given that they were still part of the warrant structure, we coded these as justifying-supporting moves, not necessarily as pedagogical proof moves.

Given our focus on pedagogical proof moves, it was more interesting to us that we also observed words describing effort (e.g., easy, not very hard, only basic algebra, simple) or apparency (e.g., of course, just, straightforward) *not* being used as placeholder warrants (17 times). Instead, they were used to describe the effort a problem would take (e.g., "I'll leave the necessary modifications as an easy exercise"), sometimes alongside an exercise for the reader, sometimes as a description of the upcoming steps. These examples appeared to serve at least one of the following two functions: (a) reassuring the reader, and (b) putting a (sub)proof into perspective. In particular, drawing on the SFL idea of a text's three metafunctions (i.e., ideational, interpersonal, and textual), we see descriptions of effort and/or apparency as possibly intending to reassure the reader (interpersonal function) as well as putting into perspective a (sub)proof so the reader knows where (not) to focus their attention (ideational function). As Halmos (1970) noted, "the reason for saying that [statements] are obvious is to put them in proper perspective for the uninitiate" (p. 137).

Last, motivated by our data, we share an observation—probably familiar to mathematics educators but novel to (undergraduate) newcomers—about the use of "trivial," a seemingly close cousin to the above words of effort and apparency. Although "trivial" can be used to indicate the difficulty of an argument, it also has its very own mathematical meaning as "the most simple example of something" (cringo, 2012) or "the canonical obvious thing" (Brett Frankel, 2012) (e.g., trivial solutions, trivial representation, trivial subgroup). By differing (slightly) from the colloquial meaning, we posit that "trivial" could be received differently by newcomers than intended (see Pimm's, 1987, writing on semantic contamination).

Adding Clarification

The third move we saw all sources engage in was *adding clarification* (20 times). We documented authors adding clarification for two main purposes:

- to add precision (e.g., "We really mean that $(\mathbf{Z}/m\mathbf{Z},+,\cdot)$ is a commutative ring."); and
- to "unpack" (or verbalize) symbols and diagrams (e.g., "(In words, T_g is just multiplication by g on the left.)").

Other Moves

There were three other moves we documented in multiple sources: (a) emphasizing (e.g., "it is the **least positive** integer") (12 times), (b) simplifying notation (e.g., "Write $g = \gcd(a,m)$") (3 times), and (c) providing extra information about mathematical objects, processes, or claims (e.g., "Define […] (This mapping is called the *natural homomorphism* from G to G/N.)") (3 times). Furthermore, we identified six moves that each appeared in only one of the sources: (a) visualizing a mathematical idea (10 times), (b) giving an example (6 times), (c) labeling an (set of) equation(s) for possible later reference (3 times), (d) amusing the reader (2 times), (e) pointing out a common student mistake (1 time), and (f) commenting on proving (1 time).

Discussion

As our current findings suggest, there is more to writing textbook proofs for pedagogical purposes than writing rigorous and complete proofs. In particular, the authors of the proofs we studied added a variety of logically unnecessary (but pedagogical) moves to their proofs—in contrast to the image of prototypical proofs. Although our focus was on pedagogical proof moves, this work also raises awareness of delimiting and justifying-supporting moves, which often intersected with pedagogical proof moves and merit further study.

As touched upon, a limitation of our work is its exclusive focus on pedagogical proof moves realized through the addition of text (as opposed to the alteration or deletion of text). Accordingly, we wonder: How could we empirically identify pedagogical proof moves in non-eliminable text? Further, we acknowledge that there are eight "other (moves)" segments that we have been unable to further describe with our current move codes. It is unclear to us if this is an issue with our identified moves or rather a natural finding about the prevalence of text segments without easily discernible purpose. Third, we recognize that although our corpus is appropriate in size for a move analysis, it has been limited to one domain of mathematics. Heeding the warning of Dawkins and Karunakaran (2016), we are in the process of expanding our corpus to Real Analysis and Linear Algebra texts, giving us an opportunity to evaluate our codes in other domains. Finally, although we have presented our work in the typical fashion of highlighting the most common codes (i.e., popular pedagogical proof moves), we wonder: Should this line of work instead highlight less common (and perhaps idiosyncratic) pedagogical proof moves to provide new ideas and possibly disrupt the status quo of (pedagogical) proof-writing?

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