A Framework for Dimensions of Student Mathematical Thinking in Instructional Planning and Teaching

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Effective teaching relies on understanding and responding to students' mathematical thinking, enabling educators to build on student prior knowledge and deepen comprehension. While extensive research has established this link at the K-12 level, the research base for undergraduate mathematics education lacks a cohesive framework that integrates student thinking into instructional practices like planning and teaching. This paper addresses this gap by proposing a comprehensive framework designed to embed student thinking more deeply into undergraduate mathematics instruction, with the goal of improving teaching effectiveness and student learning outcomes. We conclude by discussing the implications for both instructional practice and research, as well as outlining the future directions for empirical validation.

Keywords: Student Thinking, Theoretical Framework, Undergraduate Mathematics Instruction

Effective teaching hinges on the ability to understand and respond to how students conceptualize and think about mathematical ideas. This understanding allows educators to tailor their instruction in ways that build on students’ existing knowledge and foster a deeper, more robust comprehension of mathematical concepts and skills. Every teacher however, has a unique subset of beliefs that shape their planning and teaching practices (Stipek et al., 2001). These beliefs are crucial as they influence every aspect of the educational process, especially in undergraduate educational contexts (Vishnubhotla et al., 2024). Further complicating this, undergraduate instructors face constraints such as limited time for planning and developing new lessons and activities, and institutional pressures to adhere to a set syllabus that leaves little room for reform-based approaches (Johnson et al., 2013). This makes it difficult to classify particular types of instruction (e.g., inquiry, lecture) as “effective” or “more effective than the other.” Rather, we propose that we can address “effective instruction” by addressing how instructional practices are reflective of and/or consider *student mathematical thinking*.

In the complex landscape of undergraduate mathematics education, both researchers and instructors understand the importance of attending to and using student thinking during instruction (Kuster et al., 2024). Anthony and Walshaw (2009) synthesized research literature to develop a research-based “set of principles that underpin the kinds of pedagogical approaches found to develop mathematical capability and disposition within an effective learning community” (p. 148). Some examples of these principles are facilitation of classroom dialogue focused towards mathematical argumentation, building on student thinking, and selection of tools and representations to provide support for students’ thinking. Within these examples (and others provided by Anthony and Walshaw (2009)), student thinking is a major underlying theme.

Despite the critical role student thinking plays in instruction, much of the research connecting student thinking with teaching practices has predominantly focused on K-12 education, leaving a gap in the literature at the undergraduate level. Additionally, while there is a great deal of research on how student thinking can be supported during instruction, currently there is not an understanding of how student thinking unfolds in all stages of instruction nor is there consistency with the ways in which student thinking is utilized for the act of teaching by an instructor. In this theoretical report, we briefly describe how student thinking is situated in literature, and propose a framework that integrates student thinking into the planning and teaching phases of instruction.

Background Literature and Our Goal

Over the past decade, a significant body of research has emerged emphasizing the critical role of student thinking in mathematics education. For our purposes in this section, we briefly outline research in two categories; planning and in-the-moment instruction. In their discussion of planning activities that support effective instruction, Akyuz and colleagues (2013) highlight the importance of five practices: preparation, reflection, anticipation, assessment, and revision. Supporting this work, in a comprehensive literature review regarding lesson planning, anticipating students' thinking and learning was tied to better learning outcomes and improved lesson quality (Cevikbas et al., 2024). Additionally, Vale and colleagues (2019) discuss how anticipating solutions during planning, allows teachers to better elicit student thinking, and to notice and attend to variation in students' responses during instruction. Datnow and colleagues (2021) discuss how gathering evidence of student thinking continually throughout instruction is ultimately valuable for informing planning. Their work also shows how teachers can use evidence of student thinking as data, and reflect on it in conjunction with formal assessment data to identify ways they can better support student learning. This body of work provides a foundation for describing the role student thinking plays in the planning phase of instruction.

Research discussing the importance of student thinking during in-the-moment instruction has focused on specific aspects such as facilitating discussions (e.g., Leatham et al., 2015; Leatham et al., 2021; Smith & Stein, 2011), recognizing productive student ideas (e.g.,Sherin et al., 2010), engaging students in rich mathematical thinking (e.g, Henningsen & Stein; 1997; Jackson et al., 2013) and providing support to early career teachers (e.g., Amador et al., 2022; Stockero et al., 2015). Student thinking can also be thought of more concretely, as data that can be used to inform instructional decisions (Datnow et al., 2021). In line with this, the notion of *responsive teaching* focuses attention on student thinking as an implicit guide to the practice of teaching (Dyer & Sherin, 2016). Responsive (sometimes called adaptive teaching (Gallagher et al., 2022; Parsons et al., 2018; Vaughn et al., 2021)) is a student centered teaching style that involves teachers observing student thinking and adapting their teaching based on their students' needs and abilities. This premise is similar in nature to the principles of generating student ways of reasoning, building on student contributions, and developing a shared understanding from Inquiry Oriented Instruction (Kuster, et al., 2018, 2019). It is clear student thinking plays a central role in a variety of teaching practices as they relate to in-the-moment instruction.

While the current literature base forms a strong foundation, making large strides toward understanding and implementing instructional methods that focus on student thinking, it also presents a few challenges. For instance, there exists a wide array of terms that describe instructional practices related to student thinking — such as eliciting, monitoring, building on, attending to, using, leveraging, understanding, scaffolding, and interpreting — but how these terms are defined or fit together across the field is often unclear. This variation in terminology reflects the multifaceted and complicated nature of the relationship between student thinking and teaching, while also highlighting a lack of clarity and coherence across the research literature. Within undergraduate education, the application of these research concepts is even less clear. Many instructors acknowledge the importance of student thinking with regard to instruction, but in our experience struggle to systematically incorporate student thinking into their instructional practices. Moreover, there is often inconsistency in the use of these practices even across instructors that report having similar goals and using similar strategies (Kuster, et al., 2024).

Despite the communal value for using student thinking to inform instructional practices, the mathematics education community has yet to fully articulate a cohesive framework that delineates the role student thinking plays across the different aspects of the instructional process. Such a framework would ideally provide a consistent, research-based approach for integrating student thinking into instructional practices and goals, ensuring more effective teaching and improved learning experiences and outcomes in undergraduate mathematics education.

This paper aims to bridge this gap by exploring the intersection of student thinking and instructional practices in undergraduate mathematics education. We propose a framework that builds on current research to embed student thinking into every stage of the instructional process (i.e., planning to teaching). By offering a comprehensive framework and actionable insights, this paper seeks to not only contribute to the academic discourse but also empower educators to make informed, student-centered instructional decisions. When reflecting on our goals as undergraduate mathematics educators, prioritizing an environment that places student thinking at the forefront can unify the discussion regarding ‘best practice’ of teaching. Ensuring that our teaching and planning practices consistently aim to support and enhance student thinking is critical to effective teaching. Ultimately, our goal is to foster a learning environment where student thinking drives mathematical understanding and success at all educational levels.

Student Mathematical Thinking Theoretical Framework

Before discussing our framework of student thinking, we first provide our definition of Student Thinking. ***Student Mathematical Thinking*** *is any output (verbal or written) where students provide artifacts from which one can determine or infer how they thought or are thinking mathematically; it is the process of thinking, not the product from thinking.* For example, “my answer is 4” is not an example of student thinking whereas, “it cannot be 4 because that violates the axiom” is an example of student thinking. Student mathematical thinking refers to the ways in which students process, understand, and engage with mathematical concepts, problems, and reasoning. It encompasses students’ cognitive processes as they solve problems, make sense of mathematical ideas, and develop strategies for approaching tasks.

Our framework on student mathematical thinking describes the ways in which student thinking can be used by an instructor in lesson planning or teaching. Importantly, we view knowledge and learning from the emergent perspective and interpretive framework (Cobb & Yackel, 1996) where knowledge is both individual and collective. Therefore, our framework considers student mathematical thinking both in terms of the individual and collective, but we do not attempt to have our framework delineate a dimension as individual or collective as the use of student mathematical thinking while teaching and planning is both individual and collective.

The framework emerged from previous work on the lesson planning practices of mathematics faculty (Author, 2024). Our findings revealed that though faculty often aligned with a particular teaching style (e.g., inquiry-based, lecture, or a blend of both), this label did not necessarily align with their actual practices. Vishnubhotla and colleagues (2024) had similar observations. We developed this framework centered on student thinking as a means for addressing this issue by being able to characterize key aspects of instruction separate from teaching style.

In our framework we have different dimensions, each is titled with verbs that modify the phrase ‘Student Thinking’. For example, Determining Student Thinking. The framework features three dimensions within Lesson Planning, Designing, and Reflecting (Determining, Modeling, and Preparing) and five dimensions within Teaching (Eliciting, Interpreting, Scaffolding, Assembling, and Reflecting on). This framework evolved from our combined experiences as teachers and researchers across various educational settings. It reflects a fluid approach to both planning and instruction, with each closely intertwined; many in-the-moment teaching decisions are also impromptu planning actions. This framework does not adhere strictly to a linear or circular sequence, as not all steps are always executed, depending on the instructional context. With this framework it is our goal to identify foundational actions in which teachers engage, from which many other practices are composed.

Our full framework can be found in Table 1. Following Table 1 we expand upon each dimension and support it with appropriate research literature.

Table 1. Student Thinking Framework.

Category	Dimension	Definition
Lesson Planning, Designing, and Reflecting	**Determining**	Deciding on the learning outcomes of the lesson, in terms of the student mathematical thinking in which one wants the students to engage related to the topic. Note: We assume course learning objectives have already been decided and agreed upon by the university.
	Modeling	Developing possible ways in which students might mathematically think about the content at various waypoints throughout the progression of the lesson based on past experience, resources, and/or literature.
	Preparing with	Developing and/or ordering a list of possible questions, tasks, lecture points, etc. that can be utilized to elicit or progress student mathematical thinking along this (modeled) pathway.
Teaching	**Eliciting**	Engaging students in tasks, problems, and/or questions (things they do not have the answer or solution method for) with the intent of bringing their thoughts and ideas to the public space.
	Interpreting	Actively listening to students with the intent to figure out and understand their mathematical thinking and how they are using their understanding about a particular concept.
	Scaffolding	Building scaffolds that match with possible pathways students may take based on interpreted student thinking (e.g., how the students are thinking about the concepts) and how it fits with the instructional goals.
	Assembling	Having scaffolded the students' mathematical thinking might develop and lead to more sophisticated understandings, the instructor attempts to move students along that progression through careful consideration of activities that are intended to evoke current thinking in ways that lead to new and more complex understandings.
	Reflecting on	Considering how students thought about concepts and ideas, how those ideas developed throughout the lesson (for them), and how the instructor could have scaffolded questions or tasks that better aligned with that progression.

Determining Student Thinking is the initial step in using student thinking, and involves identifying a lesson's desired learning outcomes (Stephan et al., 2016; Nabb et al., 2018). This goes beyond determining the concepts to be understood and the skills that need to be mastered, and includes identifying targets related to the specific student thinking and learning one wishes to help their students develop;. To determine these goals, instructors may consult research on the topic, engage in thought experiments, or ask colleagues about their prior experiences.

Modeling Student Thinking involves making the implicit reasoning processes of students explicit by developing a blueprint for the ways students are likely to understand and reason with particular concepts or approach certain tasks. Additionally, teachers consider how students may reason with these ideas to construct understandings of more complex ideas. For instance, while students must understand and develop skills concerning limits, function, and average rate of change before they can robustly understand instantaneous rate of change, they must also think about those foundational concepts in particular ways to develop a rich and robust understanding of instantaneous rate of change, or how this can be represented as a function. In the research literature this is often referred to as anticipating student thinking (e.g., Stein et al., 2008). We have chosen the modifier *modeling,* to highlight the need for active engagement with the student thought process and the role modeling often plays in the construction process. Developing foresight into how students might understand and reason about and with a concept or idea is central to this dimension. In the research literature, modeling student thinking is emphasized as a critical aspect of lesson planning, particularly in how it informs the design of instructional tasks, the sequencing of content, and the explicit illustration of how mathematical thinking develops and can be supported (e.g., Carpenter et al., 1999; Jacobs et al., 2010; Ball et al., 2008; Peterson and Leatham, 2009).

Preparing with Student Thinking involves intentionally designing instructional activities that support students in progressing from their current levels of understanding to more advanced levels. These instructional activities (e.g, tasks, questions, lectures) are designed using the previously developed models and understanding of student thinking and learning. During the preparation phase, teachers are thinking forward and backward; forward in the sense that one must create lesson materials that support the students in engaging in the desired thinking and reasoning, and backward in the sense that one must identify lesson activities that support the thinking and reasoning that serve as waypoints to that destination. With our calculus example, teachers that use student thinking to prepare lessons are considering not only the tasks concerning the desired ways of reasoning about instantaneous rate of change, but also sequences of tasks that would support students in thinking about function, limits, and average rate of change in ways that foster student development of instantaneous rate of change. This preparation of lesson activities is discussed in the literature at both the individual lesson level (Stephan et al., 2016) and at the course level (Rasmussen & Keene, 2019).

Eliciting Student Thinking involves drawing out and making students' thought processes, ideas, and understanding observable. The goal is not to simply evaluate what students know but to engage them in deeper reflection and reasoning, helping them to articulate their thinking. Eliciting student thinking takes many forms, such as using questioning, discussions, and tasks designed to engage students in thinking about concepts (Kuster, et al., 2019). Smith and Stein (2011) discuss the importance of using cognitively demanding tasks for supporting mathematically rich student thinking and reasoning. With regard to in-the-moment decision making, teachers cannot identify productive ways to later scaffold or assemble student knowledge without first having access to their current knowledge (Hufferd-Ackles et al. 2004;

Leatham et al. 2015). Along this same vein, they cannot determine the accuracy of their models, or how to best utilize their prepared activities without having an understanding of their students' current thinking.

Interpreting Student Thinking is a critical aspect of effective instruction, as it allows teachers to understand and subsequently respond appropriately to students' conceptualizations and reasoning processes. Research underscores the importance of interpreting students' responses to adjust teaching strategies and support learning. For example, Jacobs and colleagues (2010) highlight how teachers' professional noticing—observing, interpreting, and responding to students' mathematical thinking—enables them to provide targeted support and modify instruction based on real-time insights into students' understanding. This practice is crucial for formative assessment (Franke et al., 2009), and is a key practice in Inquiry Oriented Instruction (Kuster, et al., 2019). Boaler (2016) also emphasizes that interpreting student thinking can uncover diverse problem-solving approaches and reasoning, allowing teachers to adjust tasks and discussions to better meet students' needs. By accurately interpreting students' mathematical thinking, educators can better identify where students are along the path of achieving the intended thinking goals, and what and how student ideas can be combined to support their progression along that path.

*Scaffolding Student Thinkin*g is an important aspect of in-the-moment teaching and is characterized by making decisions based on information from both the planning phase and teaching phase. That is, having interpreted the students' thinking, instructors then make decisions about how to guide and manage the lesson based on the models they developed and the materials they have prepared during the planning phase. When Scaffolding, teachers consider how students are thinking in-the-moment about a mathematical concept, and make in-the-moment teaching decisions regarding how that thinking relates to the instructional goals and how it may be utilized. They may even revise the models they developed or the materials they created in lieu of the new information they received while in the process of interpreting. Effective scaffolding requires the teacher to deeply understand students' thinking (Interpreting), anticipate how that thinking will be able to develop (Modeling), and then making instructional decisions that support that development (Franke et al., 2007; Jacobs et al., 2010).

Assembling Student Thinking can be roughly thought of as the enacting the scaffold; the actions taken by the instructor to utilize the scaffold and build the desired student thinking. This may be actualized with the utilization of a variety of instructional methods such as whole class discussions, intentionally designed task sequences, teacher-directed techniques, selecting and sequencing student presentations, etc. Effective teaching practices involve recognizing and extending students' ideas, creating productive learning environments, and promoting connections between mathematical concepts (Peterson & Leatham, 2009). The flexibility of the specific instructional methods that foster these connections is intentional, but to be Assembling Student Thinking the approach must not only incorporate students' existing knowledge, but also help them to advance their understanding through thoughtful and strategic instructional design.

Reflecting on Student Thinking is a critical aspect of effective mathematics instruction, as it plays a vital role in enhancing teaching practices and supporting student learning. By carefully analyzing students' responses and thought processes, teachers are able to make informed decisions about instructional strategies and adapt their teaching to better meet students' needs. This reflective practice helps identify where students are struggling and allows for adjustments to be made to address these challenges. Carpenter and colleagues (1996) emphasize that understanding students' cognitive processes enables teachers to align instruction with students' existing knowledge, thereby improving the effectiveness of teaching. Reflecting on student thinking provides teachers with the necessary insights to offer specific, actionable feedback that helps students build more robust and complex understandings. Additionally, Lampert (2001) suggests that this practice supports teachers' professional growth by offering insights into their own instructional practices and identifying areas for improvement.

Conclusion, Implications, and Next Steps

In this paper, we proposed a framework integrating student mathematical thinking into planning and teaching. Our framework emphasizes the importance of Determining, Modeling, and Preparing with Student Thinking during lesson planning, and Eliciting, Interpreting, Scaffolding, Assembling, and Reflecting on Student Thinking during instruction. We propose these as core building blocks from which one can characterize instructional practices across an array of pedagogical approaches. By placing student thinking at the center of instructional decision-making as opposed to teaching style, we hope to enable educators to more effectively foster deeper understanding and more meaningful engagement with mathematical concepts.

Along these lines the framework is intended to offer flexibility, recognizing that instructional practices vary widely across classrooms and teaching styles, while also providing a coherent structure for incorporating student thinking into any teaching approach. By doing so, it empowers educators and researchers to move beyond a reliance on traditional classifications of teaching methods (e.g., inquiry-based vs. lecture) and instead focus on the central goal of supporting student learning; fostering and advancing student thinking.

Despite its potential, the framework also presents several challenges and opportunities. One of the key limitations is the lack of a formal component dedicated to evaluating learning outcomes, a crucial aspect of instruction. Additionally, research is needed to explore how this framework can be adapted and applied across various instructional contexts, content areas, and student populations. We also acknowledge that our framework does not prescribe specific methods or concrete steps, but rather serves as a guiding principle. Future work should focus on empirically validating the dimensions of this framework, as well as exploring the ways in which they interact and evolve during real-time instruction.

The implications of this work are profound, particularly in its potential to unify the discourse around "best practices" in teaching mathematics. Rather than focusing on the form of instruction, we suggest that the mathematics education community should prioritize instructional practices that are steeped in student thinking. This reorientation allows for a more flexible, adaptable approach to teaching while ensuring that student thinking remains central to the instructional process, something research seems to indicate as crucial to effective teaching. As we continue to refine and test this framework, we hope it will contribute to both the research literature and practical teaching strategies, ultimately supporting educators in creating learning environments that promote deep mathematical understanding and success at all levels of education.

Future research should focus on examining how student thinking is operationalized across different types of mathematical content and instructional settings. Additionally, studies that explore how instructors in higher education can develop the skills and knowledge needed to effectively engage with student thinking will be crucial for further developing this framework. Our aim is to inspire both researchers and practitioners to place greater emphasis on student thinking, not just as a peripheral consideration, but as the driving force behind instructional planning, decision-making, and evaluation. By doing so, we can create more effective and equitable learning experiences for all students.

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