Pre-Service Teachers' Strategic Noticing During Modeling Through Centers of Focus

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Mathematical modeling tasks require modelers to consider both real-world conditions and mathematical properties while constructing models that represent the given real-world phenomenon. While the field is matured with research on the processes involved in constructing models, little research has explored what modelers center their focus on (i.e., notice) while solving modeling problems and how those centers of focus inform the strategies for solving modeling tasks. Through analyzing data from two pairs of undergraduate pre-service teachers' work on modeling tasks, we found three cases in which PSTs' centers of focus informed their strategies for model construction. In this report, we illustrate two of those cases and discuss implications for research and instruction on modeling.

Keywords: modeling-oriented noticing, mathematical modeling, strategic noticing

To enhance student experiences in mathematical modeling, the mathematical modeling education community has undertaken numerous research programs. For example, scholars have conducted research on student thinking in modeling (e.g., Borromeo-Ferri, 2010), teacher knowledge for modeling (e.g., Greenfrath et al., 2022), scaffolding strategies that advance students in their modeling process (e.g., Czocher et al., 2024) and designing modeling tasks and learning environments through employing theoretical frameworks (e.g., Lesh and Doerr, 2003). The majority of these works have been built on a simplified framework of the modeling process, commonly referred to as modeling cycles or modeling process diagrams (e.g., Blum and Leiß, 2007; Niss 2010). From the cognitive perspective of modeling, a modeling cycle usually consists of a real-world domain, a mathematical domain, and modeling competencies that are unique to solving modeling tasks (usually depicted as transition phases from one domain to another or within domains). Examples of micro-level modeling competencies include systematizing, mathematizing, validating, interpreting, and understanding the situation (e.g., Blum and Leiß, 2007). Examples of global competencies include metacognition, reflection, and anticipation (e.g., Stillman, 2011). In some modeling cycles, there is a clear separation between the real-world domain and the mathematical domain (e.g., Niss, 2010), while in others, these domains are interwoven (e.g., Czocher, 2018). Regardless of the extent to which the real-world domain and the mathematical domain are integrated, Stillman and Brown (2014) argue, the identification of the two domains as distinct is based on the epistemological foundation that real-world and the mathematical world "co-construct each other" (Doerr and Pratt, 2008, p. 260). This separation between the two domains, is at the "heart of modeling" (Stillman and Brown, 2014, p. 3). This is because the modeler has to *notice*—a global competency—both real-world conditions and mathematical elements and transition between these domains simultaneously (Blum, 2011) to construct a model that is viable to represent the experienced phenomena. However, little research has been conducted on how modelers' noticings of real-world and mathematical properties enable or constrain their modeling process. Knowing what students notice and how they attend to what they notice is instrumental for designing modeling tasks that can elicit certain modeling competencies and for contingent scaffolding—skills necessary for teaching modeling. In light of this, the objective of this paper is to shed light on learners' modeling-oriented noticing as they constructed models to represent real-world situations.

Background and Theoretical Framework

Noticing refers to intentionally focusing on particular phenomena so as to inform one's decision-making in the future (Mason, 2008). Noticing became popularized within the field of education through the domain of "teacher noticing" (Jacobs et al., 2010; Sherin & van Es, 2005). Teachers notice phenomena in their classroom, and these noticings inform their decision-making. There are three elements of teacher noticing: attending, interpreting, and deciding how to respond (Jacobs et al., 2010). First, teachers attend to aspects of students' mathematical strategies and interpret students' mathematical understanding based on those "attendings." Then, teachers make decisions about how to respond to students based on their interpretation of students' mathematical understanding. Scholars have explored *what* teachers notice, *how* they notice, and productive ways to notice. Within the field of mathematical modeling, among other competencies required for teaching modeling (For example, see Borromeo Ferri and Blum, 2011), modeling-specific noticing has been explored as an avenue to support students' learning of modeling. For example, Alwast and Vorhölter (2022) conceptualized teacher noticing within modeling as teachers' skills in perceiving, interpreting, and decision making based on students' difficulties, metacognitive strategies, and approaches to solving modeling task (pp. 167-268). On the other hand, Glabraith (2015) identified three classes of modeling specific noticing: Noticing as a mathematician (determining the situations that can be modelled using mathematics), noticing as an educator (designing tasks that support modeling competencies), and noticing as a student mentor (noticing students' in-the-moment modeling-oriented thinking) (p. 11).

Recently, scholars have expanded the noticing literature to examine how and what students notice to inform their mathematical thinking (Campbell & Yeo, 2023; Hohensee, 2016; Lobato et al., 2012; Lobato et al., 2013). In this study, we build upon recent literature on students' modeling-oriented noticing (Galbraith et al., 2017), that extends Galbraith's (2015) classes of modeling-specific noticing to also include *noticing as a student*. Here, the focus is on the specificities of what students notice as relevant in order to progress in constructing models of real-world situations. Galbraith et al. (2017) included modeling-oriented noticing as a component of anticipatory metacognition, a global modeling competency, which "shifts the emphasis towards reflection that points forward to actions not yet undertaken (p. 84)." The authors argue that successful modeling not only requires 'noticing' but also discerning the relevance of what has been observed (p. 85). Therefore, the authors defined a new construct, productive modelling-oriented noticing as, "the process of modelers noticing in a productive manner what is important and what is not in order to generate strategies" for successfully carrying out a modeling activity (p.84). Within modeling-oriented student noticing, Galbraith et al. (2017) proposed two ways in which noticing can be expressed: strategic noticing and explanatory noticing. We interpret strategic noticing as noticing the steps required, including macro and micro-level steps, to achieve modelers' modeling goals (e.g., decision to include air-resistance to model the speed of a falling ball). We interpret explanatory noticing as noticing that serves as an explanation of their approach to solving the task (e.g., because air-resistance can slow the speed of a ball). Since the strategies generated inform modelers' next steps, it can be said that *strategic noticing* is a key driver of modelers' progress through the modeling process.

The cognitive processes of sorting relevant information from competing information are not new; it has long been used to study how individuals construct knowledge (e.g., Piaget, 1952). Typically, what surfaces as relevant from the rest are shaped by the goals the learners set for themselves (Simon et al., 2004; von Glasersfeld, 1995). Lobato et al., (2012) refer to these significant aspects of noticing as 'centers of focus,' defining them as "features, regularities, properties, or conceptual objects to which individual students pay attention" (p.439). Within a mathematical modeling context, where the modeler is required to transition between the real-world and mathematical domains simultaneously, either the real-world conditions, the mathematical elements, or both become the centers of focus that inform the modelers' strategies. Therefore, in this report, we address the following research question: *How do real-world and mathematical centers of focus inform modelers' strategic noticing*?

Methods

Data were collected by conducting task-based interviews (Goldin, 1997) with two pairs of undergraduate pre-service teachers (PST) enrolled in a public university in the western United States. The first pair consisted of Gwen and Arya and the second pair consisted of Luna and Sahana, with all names being pseudonyms. We chose to conduct the interviews in pairs to create an environment conducive to observing PSTs' modeling-oriented noticing due to their communication with each other as they co-constructed mathematical models. Each pair participated in 8 interview sessions. In the first 7 sessions, the PSTs worked on 11 mathematical modeling tasks, which were adapted from prior literature. These tasks were selected due to their heterogeneity of modeling characteristics (see Maaß, 2010) and their appeal to socio-justice and environmental issues. The last session was used for reflection and member-checking. In this paper, we report on Luna's and Sahana's work on the tasks shown in Table 1.

Table 1. Tasks from which findings are reported

Tasks	Description (abridged versions)
Hospital Task (Zbiek and Connor, 2006)	Boise (Idaho), Helena (Montana), and Salt Lake City (Utah) are three large cities in the northwestern part of the United States. The map shows the locations of the three cities as well as other large cities in the area. Suppose you are hired to determine the best location for the hospital. Where would you locate the hospital?
Piano Tuners Task	How many piano tuners are there in the city of New York?

The interviews were conducted in a laboratory setting where a video camera was placed to record PSTs' written work and gestures. The PSTs were given freedom to interpret the tasks in any way they chose, and they were not given any time restrictions for completing the tasks. The interviewer's questioning mainly focused on unpacking PSTs' reasoning and uncovering the aspects of PSTs' modeling-oriented noticing. For example, to evoke justification of their strategies, the interviewer asked questions such as "why was it important to consider this?" "why did you make that assumption?," and "how reasonable is the assumption that you made?." The interviewer maintained an ongoing research journal where she recorded her reflections and possible conjectures of PSTs' reasoning at the end of each interview.

Since the overarching goal of the study was to investigate PSTs' modeling-oriented noticing, we used an a priori framework for our initial coding. Two members of the research team coded the videos for strategic noticing (Galbraith et al., 2017), where the unit of analysis was chunks of text that discuss a single idea. Table 2 provides a list of indicators that were developed using techniques from constant comparison methodology (Strauss and Corbin, 1990) to identify PSTs' strategic noticing. The research team met and discussed any discrepancies in the coding until we reached complete agreement.

Finally, each instance of the strategic noticing was coded for real-world and mathematical centers of focus. If the strategic noticing was explicitly grounded in real-world considerations, it was coded as real-world. If the strategic noticing was based on information given in the task, simplification for mathematical convenience, mathematical strategies (i.e., writing an equation, finding a quantity to proceed), and mathematical properties, those instances were coded as mathematical center of focus. The explanations and justifications the PSTs provided for their strategies were used to determine the centers of focus.

Table 2. Indicators for strategic noticing

Indicator	Definition
I1: Taking stock	PSTs review key points through *reiterating*, *underlining/circling/highlighting,* or *recalling* previously provided information from the modeling task.
I2: Researching information	PSTs search and gather relevant data and information through technology.
I3: Identifying parameters	PSTs identify specific parameters or variables necessary to progress
I4: Identifying the need to make assumptions	PSTs acknowledge and express the need for making assumptions to progress.
I5: Simplifying the situation	PSTs simplify complex situations, making model progression possible. Examples include making necessary assumptions, prioritizing and sorting the most critical assumptions from less important ones, rounding values and/or purely guessing the values of a parameter.
I6: Proposing strategies and considerations	PSTs brainstorm and consider various approaches, strategies, and factors for developing and solving the model. This phase often involves uncertainty or hesitation by students as they deliberate on an approach to adopt.
I7: Implementation of a strategy	PSTs execute the planned steps or approach for constructing a model by breaking the problem into smaller steps, assigning known values to parameters, and making decisions for choosing certain computations. This phase typically involves greater certainty in the adopted approach and less hedging or hesitation.
I8: Verifying/ validating the model or approach	PSTs check the accuracy, reliability, and effectiveness of the implemented model or approach. This includes ensuring that the model's outcomes align with expected results, testing against known data or information, and validating assumptions and calculations.
I9: Noticing patterns and characteristics of the model	PSTs observe and recognize patterns, mathematical properties, or features inherent in the model. Examples include identifying relationships between variables, observing trends in data, or noticing underlying principles that affect the model's behavior

Findings

We present our findings through three cases: (1) Noticing of mathematical elements foregrounded; (2) Noticing of real-world conditions foregrounded; and (3) Noticing of both mathematical elements and real-world conditions foregrounded. In case 1, PSTs' strategic noticing was informed primarily through mathematical elements as their centers of focus. In case 2, PSTs' strategic noticing was informed through real-world conditions as their centers of focus. That is, the centers of focus driving each strategy were grounded in real-world conditions. In case 3, PSTs strategic noticing was informed either by real-world conditions, or mathematical elements, or both. Due to space constraints, we illustrate two of the cases below. In our illustrations, we identify the strategic noticing (**I#**), and the centers of focus for each strategic noticing (**R** for real-world, **M** for mathematical elements). Concluding each case, we describe how the students' centers of focus informed their modeling strategy.

Case 2: Noticing of Real-World Conditions Foregrounded

We illustrate this case using Luna and Sahana's work on the hospital task. After reading the task, Luna suggested that she wanted to connect the three cities and make a triangle and proposed that the hospital would be somewhere within the triangle (**I6, R**). She made this proposal because she wanted to speculate which other cities would also use the hospital, justifying her strategy based on real-world considerations. Luna justified her strategy as:

> Luna: If it's the three points, it's the areas surrounding it. A lot of areas in Utah especially that's shown here, has a high population. Montana's a bit more spread out. People from Wyoming might be coming, depending on where it is. Northeastern Idaho is the more populated area of Idaho…So I'm just thinking this is (referring to her triangle), it just gets a rough estimate of where there's more people and who all is going to that hospital.

Next, Sahana expressed her dilemma in choosing a hospital location near the I-15 freeway (**I6, I1, R**) for better accessibility versus choosing a location based on the population in the major cities and surrounding areas (**I6, R**) because she stated more population implies more use of the hospital. Luna suggested that Salt Lake and Boise are more concentrated therefore it would make sense to put the hospital closer to Boise and Salt Lake area, for example either in Idaho Falls or Pocatello, rather than closer to Helena (**I6, R**), once again prioritizing real-world conditions.

The interviewer gave additional information regarding the population of each city. The data given was consistent with PSTs' conception of the population of the three cities. The PSTs decided that the hospital should be placed either at Pocatello or Rexburg (**I6, R**) because the two cities are closer to Salt Lake City and are on the highway, therefore easily accessible, justifying their proposal through real-world conditions.

Based on her personal experience living in Pocatello, Luna suggested Pocatello as a great place for hospital (**I6, R**) because it is the junction for many other popular highways (such as I-86 interchange). Sahana concurred as well. Both PSTs agreed that Pocatello was the next best city for a hospital after Salt Lake City and drew a star on the map to finalize their solution. Furthermore, the PSTs agreed that Pocatello was a good city for a hospital as evidenced in the exchange below:

> *Luna:* We want to go where there's a higher concentration of people because that makes it more accessible if there's more people. Pocatello has, it's just where all of the major roads or highways and interstates combine. And so it's just a good meeting spot.
>
> *Sahana:* …Yeah. Realistically Pocatello is a better place to be equipped for building a hospital, funding a hospital, maintaining a hospital.
>
> *Luna:* Yeah. Staff in the hospital. Yeah.
>
> *Sahana:* Relative to main roads. Real world ability.
>
> *Luna:* It's like the ability to support or community's ability to support… a hospital. To be able to staff it and maintain it. And it has to be in a place that it's also going to draw people to want to work there…
>
> *Sahana:* And live there.

In this vignette, we presented six instances of strategic noticing, all of which were informed by PSTs real-world considerations as their centers of focus. That is, the PSTs approached the modeling task by strategically by prioritizing only real-world conditions (over mathematical elements) by attending to the following: the population of each city, highway accessibility, the city's capacity to fund and maintain a hospital, and its ability to attract people to relocate for work.

Case 3: Noticing of Mathematical and Real-World Elements Foregrounded

To illustrate this case, we use Luna and Sahana's work from the Piano Tuners task. After reading the task, Sahana said that they need to make assumptions (**I4, M**) because "we don't know anything, so we literally, we're going to fill it," addressing the scope of the task. The PSTs suggested that they need to identify the number of people in New York city (NYC), and how many of those people have pianos (**I3, M**). They further proposed that whether a person plays a piano or not would not affect their model (**I5, I6, R**) because "some people will just get it tuned", "some people play the piano and don't get it tuned", and that "some people get it tuned way after they should have. It's not a consistent amount of time." Here, the PSTs were justifying their approach through real-world considerations. The PSTs researched the population of New York as 8.468 million, which they approximated to 8.5 (**I5, M**).

Next, Luna suggested that there were not very many pianos in NYC that need tuning (**I6, R**) because NYC apartments are small. Luna further suggested that that there are probably more electric keyboards as they are easier to move into smaller apartments, justifying her approach through real-world considerations. Therefore, the PSTs assumed that there were 956,000 pianos in total (those that need tuning and those that do not) (**I5, R, M**). Sahana commented that 956,000 seemed like a lot of pianos but expressed that she can commit to it (**I7, R**) since "they (NYC) do music-y things", have "big college that does music stuff", and that there is Broadway, where her commitment was informed through her noticing of real-world considerations.

Next, The PSTs wanted to determine "how many of them [pianos] need to actually be tuned?" (**I3, R, M**). Here, the PSTs decision to consider only a portion of the pianos as needing to be tuned was informed through their real-world knowledge and their mathematical goal towards estimating the number of piano tuners. The PSTs assumed "one third of all pianos are classic, regular, pianos [that need tuning]" (**I5, R, M**) and evaluated the number of piano tuners that need tuning as 315,480. The PSTs proceeded with determining "how many pianos can a tuner tune in a year?" (**I3, M**). To accomplish this, Sahana simplified the situation (**I5, M**) by assuming that all pianos need to be tuned once a year. She commented on the reasonableness of the assumption as "I don't think they actually do, but we're just going to do it for our sake," indicating her decision was driven primarily for mathematizing purposes. Sahana proposed that a piano tuner can tune a piano in 30 mins (**I6, R**). This decision was based on two real-world considerations: "piano tuners are insanely good. They can do it really quick" and "you have to factor in travel time." Next, the PSTs assumed (**I5, R**) that a piano tuner's work week consists of 5, 8-hour workdays. The PSTs justified their assumption as, "he's a busy piano tuner, got to put food in the table." Given these conditions, the PSTs computed the number of pianos tuned by a piano tuner during a week as 80 (see Figure 1).

Having determined the number of pianos tuned in a week, the PSTs proceeded with considering the number of piano tuners tuned in a year (**I3, M, R**). While the PSTs assumed that there were 54 weeks in a year (**I5, R**), Luna then brought up that the piano tuners will not be working all 54 weeks (**I6, R**) because piano tuners are "probably not going to work on Christmas." The PSTs decided that there 52 working weeks within a year (**I7, R**). Under these assumptions, the PSTs found that a piano tuner can tune 4,160 pianos in a year (see Figure 1). Finally, Luna proposed (**I5, M**) taking "how many pianos there are (315,480), and divide[ing] it by 4,160 (i.e. number of pianos needing tuning per year) [to] give the total number of piano tuners”. The result was 75.8, which the PSTs approximated to 76 (**I5, R**) because the PSTs argued that there can’t be partial piano tuners. However, the PSTs proposed (**I6, R**) that 75.8 could be accounted for by the fact that some piano tuners may work part-time.

In this vignette, we presented 20 instances of strategic noticing. In particular, to find the number of piano tuners in NYC, the PSTs noticed that the following parameters first need to be determined: the population of NYC, the number of pianos in NYC, the number of pianos need tuning, and how many pianos can a piano tuner tune in a year. Each of these parameters were consequences of the PSTs considering both mathematical centers of focus (building on prior quantities to arrive at the final quantity) and real-world centers of focus (piano tuners do not work during Christmas). As illustrated in the vignette above, there were some strategies, where the centers of focus were both math and real-world elements which primarily served the PSTs to consider a new quantity as a result of a real-world constraint (for example only considering a subset of the number of pianos because not all pianos need to be tuned).

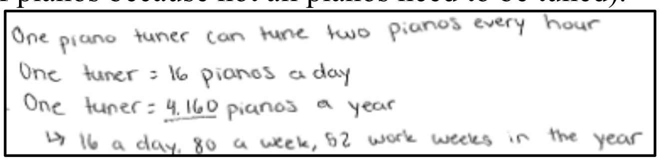

Figure 1. Luna and Sahana’s work for the piano tuners task

Discussion

Our paper has three primary contributions. First, we contribute to the existing research on PSTs and modeling by positioning PSTs as *learners of modeling* through a novel cognitive lens, rather than focusing solely on typical modeling competencies. Second, our findings have important implications for design and implementation of modeling tasks. For example, in the piano tuners task, the PSTs considered both real-world conditions and mathematical strategies when developing a model for the situation, at times alternatively and at times simultaneously. The characteristics of PSTs’ centers of focus may be attributed to the nature of the piano tuners task, which was open ended, artificial, and therefore allowed for more creativity. In contrast, in the more realistic hospital task, the PSTs prioritized real-world conditions over the use of mathematical strategies (e.g., measuring distance proportional to population size). This may be attributed to PSTs’ local knowledge about the cities and the surrounding areas. Therefore, we ask: *How real do we want modeling tasks to be if we want student to develop certain modeling competencies such as using mathematical approaches to solving a modeling task*? Finally, our study contributes theoretically and methodologically to existing frameworks on *noticing*. Theoretically, we build on noticing as a teacher competency to also include *noticing as a student* specific to modeling. Our findings can be used to make inferences about what students attend to, how they interpret those attendings, and what decision they make based on their interpretations depending on students’ centers of focus. For example, when the center of focus is real-world conditions, students attend to the real-world conditions that may affect the model (*attending*), analyze the perceived information based on personal or real-world experience (*interpreting*), and place emphasis on the significance of the real-world conditions over mathematical strategies in model development (*responding*). Methodologically, we generated a list of indicators that can be used to identify strategic noticing (Galbraith et al., 2017). Research within the cognitive perspective of modeling has largely stagnated over the last decade, where the emphasis is placed on phases of the modeling process (such as those depicted in the modeling cycles). Following Galbraith et al (2017), we see our work as advancing the modeling field in a newer direction through introducing a fresh perspective for analyzing student thinking in modeling.

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