Conceptual Chunking and Algebraic Proofs

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Research in cognitive psychology identifies chunking as one of the driving information-processing mechanisms in human cognition. Conceptual chunking, a term specific to Halford's relational complexity theory, provides theoretical insights into what constitutes chunking. Specifically, conceptual chunking explains the chunk formation as the result of temporal suppression of relations between the items within the chunk. In this article, I present the analysis of two undergraduate students' proofs from the relational complexity theoretical perspective. Preliminary results suggest that conceptual chunking can help describe students' mathematics and explain some of their difficulties related to proving algebraic conjectures.

Keywords: cognition, theory of relational complexity, chunking, proofs

Beginning in the 1950s, studies in cognitive sciences have revealed substantial evidence for a unifying information-processing mechanism known as *chunking*. In his foundational paper, Miller (1956) described chunking as combining several units of information into a single familiar unit or a chunk. Miller proposed that chunking differs from *grouping* in that the former requires the chunks to correspond to some pre-existing representations. For example, the sequence of 12 seemingly unrelated letters AMIRBRUMEBBC can be chunked into four cognitive units of varying sizes [AM][IRB][RUME][BBC] by identifying familiar words and acronyms. Similarly, Cowan (2001) defined chunks as groups of items with strong, pre-existing associations with each other but weak associations with other items and chunks.

Since Miller's work, researchers have established chunking as one of the key information-processing mechanisms in various aspects of human cognition, including mathematical reasoning (e.g., Harel & Kaput, 2002; Schoenfeld, 2001). Several influential theories in mathematics education are reminiscent of the idea of chunking, such as *encapsulation* (Ayers et al., 1988) or *reification* (Sfard, 1989). With some flexibility of interpretation, chunking can be viewed as an instance of *reflective abstraction* (Beth & Piaget, 1966), in which "a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower" (p. 247).

Despite the recognized importance of various forms of chunking for mathematical reasoning, little is known about the underlying cognitive mechanism. The construct of *conceptual chunking* provides a theoretical explanation for how chunking takes place. Proponents of the relational complexity theory (Andrews & Halford, 2002; Halford et al., 1998) suggest that conceptual chunking involves temporal suppression of the relations between the items within a chunk. Principles embodied in Halford's theory have been validated in several empirical studies (Andrews & Halford, 2002; Birney & Halford, 2002; Halford et al., 2005; Waltz et al., 1999); however, they were never explored in the context of undergraduate mathematics, and, in particular, proof production.

This preliminary study aims to investigate the psychological construct of conceptual chunking in the context of mathematical proofs. The overarching research question is *What are the aspects of conceptual chunking in proving algebraic conjectures?* In this report, I first elaborate on Halford's relational complexity theory and then apply this framework to analyze two undergraduate students' ways of producing an algebraic proof. Preliminary results suggest that conceptual chunking can be an effective way to model students' mathematical reasoning and explain some of the difficulties related to proving algebraic conjectures.

Framework: Relational Complexity Theory and Conceptual Chunking

Halford and colleagues (e.g., Andrews & Halford, 2002; Halford et al., 1998) proposed that the cognitive complexity of a task should be defined not only in terms of the number of units of information it requires to be coordinated but also in terms of the complexity of relations between them. Consider, for example, the following problem: *Suppose five days after the day before yesterday is Friday. What day of the week is tomorrow?* (Sweller, 1993). This problem may seem to appear frustratingly difficult because of the necessity to establish relations between numerous elements presented in the first sentence, which cannot be considered meaningfully in isolation. As such, the cognitive task becomes more complex when the number of relations that need to be maintained in parallel increases.

The complexity of the task can be reduced via the mechanisms known as segmentation and conceptual chunking. Segmentation entails the decomposition of the task into smaller segments or steps that can be processed serially. Examples of segmentation include breaking the proofs into cases or implementing proof by induction as a three-step procedure (i.e., the base case, inductive hypothesis, and inductive implication).

Conceptual chunking, the construct that is central to Halford's theory and this study, entails combining units of information into a single chunk at the cost of making the relations between the units temporarily inaccessible. Halford illustrates this construct using the concept of velocity (e.g., Halford & McCredden, 1998; Halford et al., 1998). Defined as the ratio of distance over time, $V=S/t$, this concept involves the interaction of three variables and can be viewed as a three-dimensional representation. It is also possible to think of velocity as the number displayed on the speedometer $V=60$ km/h – a single-dimensional representation. The transition from the three-dimensional to the single-dimensional representations makes relations between the velocity, distance, and time temporarily inaccessible. Chunked concepts can be further combined with other chunks to construct higher-level concepts. Thus, velocity can be used to define acceleration, which can be used to define the notion of force $F=ma$. However, if the task requires computing the way velocity varies as the distance doubles and time does not change, the relations between the variables must be restored.

To summarize, conceptual chunking helps students to make progress on the task by temporally suppressing the relations between different cognitive units. Chunking is a matter of expertise (Chase & Simon, 1973) and can be viewed as "a form of learning that takes place over time" (Halford et al., 1998, p. 810). The three general principles of conceptual chunking are

1. A chunk functions as a single entity.
2. No relations can be represented between items within a chunk.
3. Relations between the chunk and other items, or other chunks, can be represented.

Methodology

The study draws on the data collected for a larger project investigating the role of students' gestures in offloading cognitive demands on their working memory during various proving activities, such as reading, presenting, and constructing proofs of mathematical conjectures. The participants were undergraduate students enrolled in an introductory proofs course. The students participated in several individual task-based interviews lasting approximately 45 minutes each. The participants were asked to think aloud as they worked with proofs. Because the goal of the overarching project was to study students' gestures, writing and drawing were not permitted.

In this paper, I report on two students, Ella and Sherry, proving the conjecture known as a Diophantine Equation (Figure 1).

Prove that there are no integers x and y such that $x^2 = 4y+2$

Figure 1. Interview task.

One way to prove it is to observe that the right side of the equation is even. Therefore, the left side also has to be even. Moreover, because the left side is a perfect square, then it has to be divisible by four. However, the right side is not divisible by four, and thus, no integers x and y satisfy this equation.

At a later date, a stimulated recall interview (SRI) was conducted. During this interview, the students were shown selected video clips of themselves working on the task, and the interviewer asked questions about their mathematical reasoning. All interviews were video and audio-recorded.

Data Analysis

The analysis began with creating the transcripts of the interviews and identifying the episodes where the participants seemed to chunk mathematical objects involved in their proofs. Verbal evidence of chunking included labeling pieces of the equation as "it," "a fraction," "number," etc. Students' responses during the SRI were used to triangulate the initial findings. One of the goals of this preliminary report is to identify more rigorous data analysis procedures for future study design.

Results

In this section, I present the analysis of students' proofs from Halford's relational complexity theory perspective. Specifically, I showcase two episodes of conceptual chunking and illustrate its underlying cognitive mechanisms in constructing algebraic proofs.

Sherry's Proof

Sherry made two attempts to prove the given conjecture. In the first attempt, the student relied on the fact that the existence of rational solutions would entail the non-existence of integer solutions. After some discussion with the interviewer, she realized that her initial approach was invalid. On her second attempt, Sherry started her proof by solving for y and rearranging the equation as $y = \frac{x^2}{4} - \frac{1}{2}$. However, later on, the denominator of the fraction $\frac{x^2}{4}$ was lost, and the equation was transformed into $y = x^2 - \frac{1}{2}$:

> So, to prove that there are no integers x and y such that x squared equals to 4y plus two... Um... I think... I'll think of cases, where x and y are integers. And see if x squared equals to 4y plus two exist. And thus, I rearrange the equation... in y... so x squared minus two equals to 4y, and y equals to x squared over four minus two over four. And since the very simplified rational number of two over four, that's one over two, **we have y equals to x squared minus one over two**. And since we have assumed that x and y are both integers, **x squared is also going to be integers. But then integer minus one over two is just the... em... very simplified version of rational numbers** [i.e., not an integer]. And it will also give a rational

number... x squared equals to four... There are no integers x and y, such that x squared equals to $4y$ plus two.

Initially, the participant correctly solved for y and derived the equation $y = \frac{x^2}{4} - \frac{2}{4}$. Then, she shifted the focus of her attention to simplifying the second fraction from the right side from $\frac{2}{4}$ to $\frac{1}{2}$. In doing so, the student chunked the components of the first fraction, $\frac{x^2}{4}$, into a single cognitive unit by suppressing the relations between the numerator and the denominator of the fraction. However, the reconstruction of these relations turned out to be problematic and significantly affected Sherry's proof.

Ella's Proof

Ella's approach was based on the valid idea of factoring out 2 from the right side of the equation and observing that $2y+1$ is odd, and therefore, $2(2y+1)$ is not divisible by 4. However, the interviewer asked her to summarize her proof, Ella lost the coefficient 2 in front of y, which implied that nothing can be specified regarding the parity of the quantity enclosed in parentheses:

> We're gonna try to prove that there are no integers x and y such that x squared equals $4y$ plus two. So first, we're going to assume that x and y are integers, in which case, x squared becomes a perfect square number. However, $4y$ plus two, when you pull it two out, **you can factor the greatest common factor of 4*y* plus two is two, which gives you two parentheses *y* plus one because we know that *y*... Wait... shoot... We don't know anything about *y* being even or odd... Em... Here we have cases then...** If y is even, then whatever's in the parentheses is an odd number, which means that there is no way that it can be a perfect square or equal to the perfect square of x squared. Because there's a two factor that's not matched... within the other factors... to be able to... to take the square root of... in the case that y is even, the whatever's in the parentheses... sorry, in the case that y is odd, then whatever's in the parentheses will be an even number if it's plus one.... **Oh my gosh. I forgot about the 2*y*.** Can I jump back into my previous [laughing]? Okay, that's what makes it even. That's what I was missing.

Making progress on the task, the participant chunked $2y$ and $+1$ into a single unit signifying an odd number. Cognitively, this could be done by suppressing the relations between 2, y, and 1. However, the details about why this number is odd were temporarily lost. This can be explained by Ella's difficulty in reconstructing the relations between 2 and y. As a result, $2y$ was replaced by y, transforming the expression $2y+1$ into $y+1$, which changed Ella's proof. The observed effect of conceptual chunking was confirmed later during her SRI:

> *Researcher*: I find it interesting that, in the middle of your presentation, you decided that the parity of y might affect the proof. You had to consider two cases separately: y is even and y is odd.
>
> *Ella*: Right, because I was saying y plus one. Because in my head, it was $2y$ plus two. For some reason, I was looking at the four. Um... Because I think... when I think about, like, the structure of an odd number, right? They give us a... there's, like, $2k$ plus one, right? Where k is some sort of the whole number. Um... Because you know that that $2k$ is going to be even and then the other part plus one makes it odd, right? I think of the $2k$...

together. Like I don't think of it as two times *k*, it's just 2*k*. So, I think that's where I was like **instead of 2*k* I just put *y* there, and why I forgot about that 2*y***. And so I was, like, well, then I could save the inside of that two times whatever's in the inside, *y* plus one. Cause that, they're gonna be even or odd. So, I started going into cases, randomly.

Discussion

Theories of working memory capacity (Baddeley & Hitch, 1974; Pascual-Leone, 1970) suggest that humans can operate on a limited number of chunks at the same time. According to the principle of cognitive economy (Rescher, 1989), we tend to choose the least complex representation available to make progress on a cognitively demanding task. Thus, it is the nature of human cognition to package information into chunks by reducing the complexity of relations between the units.

The study design enabled my participants to coordinate the enormous amount of information and mental representations when constructing the proof of the given algebraic conjecture. The data presented in this study indicate that Sherry and Ella implemented mental strategies that can be described as conceptual chunking. Sherry temporarily suppressed the relations between the numerator and the denominator in the fraction $\frac{x^2}{4}$. Ella temporarily suppressed the relations between 2, *y,* and 1 in the expression $2y+1$. Although conceptual chunking did help the students to make progress on the cognitively demanding task, the reconstruction of the suppressed relations turned out to be problematic in both cases. Sherry overlooked the denominator of the fraction, and Ella lost the coefficient of 2. The described negative effect of conceptual chunking, as applied to mathematical reasoning, is a finding not previously reported in the literature. More research is needed to investigate this preliminary observation.

Questions for the Audience

I conclude my report with the following three questions for the interested audience:

1. Is conceptual chunking and the overall relational complexity theory appropriate theoretical perspective for explaining the observed phenomenon?
2. What would be an appropriate study design for future research on conceptual chunking?
3. What are the pedagogical implications of these preliminary results?

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