When Different Conceptions Hold Together: A Case of a Student's Coordinate Systems and Graph Reasoning

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We report on a qualitative case study of a college algebra student, Jamie, who exhibited multiple conceptions of coordinate systems (Lee et al., 2020) and forms of graph reasoning (Johnson et al., 2020) during a single task-based interview. Our analysis of Jamie's graph sketches and interactions with digital tools during the interview highlighted connections between her quantitative reasoning and conceptions of graphs. Specifically, Jamie's conception of a quantitative coordinate system was closely linked to her reasoning about changing attributes and their relationships, whereas her conception of a spatial coordinate system was closely linked to her reasoning about the physical motion within the dynamic situation. Jamie's case speaks to the complexity of students' quantitative reasoning and illustrates the interconnectedness of their conceptions of coordinate systems and graph reasoning.

Keywords: undergraduate education, technology, quantitative reasoning, graphing conceptions

Sketching graphs is a productive way for students to represent and reason about relationships between attributes in dynamic situations (Carlson et al., 2002; Thompson & Carlson, 2017). However, the effectiveness of this activity is closely tied to students' underlying conceptions of coordinate systems (Lee et al., 2020) and their graph reasoning (Johnson et al., 2020). Emerging research suggests a potential interconnection between these constructs (Johnson et al., 2024; Paoletti et al., 2022). Our goal for this qualitative case study (Stake, 2000) was to analyze a college algebra student's conceptions of coordinate systems alongside their graph reasoning. This study seeks to delve deeper into the relationship between these constructs by exploring the question: What is the nature of students' conceptions of coordinate systems and graph reasoning within digital graphing tasks involving dynamic situations?

Background

Quantitative and Covariational Reasoning

To frame our study, we draw on Thompson's (1994, 2011) theory of quantitative reasoning. According to Thompson, students conceive of attributes of objects as quantities when they consider them to be measurable. When students conceive of relationships between attributes that are capable of varying simultaneously with one another, they are engaging in covariational reasoning (Thompson & Carlson, 2017). For example, consider a situation in which a toy car travels at a constant speed around a traffic cone (Figure 1). Given this dynamic situation, a student may conceive of the toy car's total distance traveled and distance from the cone as attributes that are possible to measure, thereby conceiving of them as quantities. If the student were to conceive of a relationship between these attributes in which they vary simultaneously with one another, then they would be engaging in covariational reasoning.

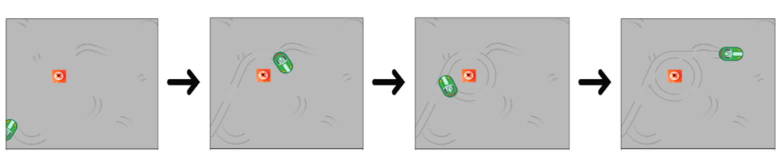

Figure 1. Toy Car Dynamic Situation.

Spatial and Quantitative Coordinate Systems

Lee et al. (2020) distinguished between two kinds of coordinate systems that students may conceive of when representing a situation: spatial and quantitative. A student conceives of a *spatial coordinate system* when they mentally overlay a coordinate system onto a physical or imagined space to determine relative locations within that space. For example, given the dynamic situation in Figure 1, a student may conceive of a spatial coordinate system by imagining a collection of points representing the toy car's position at different moments in time and sketch a graph resembling the toy car's path through the physical space.

A student conceives of a *quantitative coordinate system* when they extract and coordinate quantities from a situation to represent the quantities and the relationship between them in a new space, such as a Cartesian plane. For example, given the same situation, another student may disembed the toy car's distance traveled and distance from the cone, then plot these quantities on a graph with total distance traveled represented on the x-axis and distance from the cone represented on the y-axis. In doing so, the resulting graph would be representing the relationship between the two quantities. Lee et al.'s (2020) distinction offers a framework for analyzing students' conceptions of graphical representations in various contexts.

Graph Reasoning

Johnson et al. (2020) developed a graph reasoning framework to characterize students' conceptions of graphs. The framework includes four broad types of reasoning that encompass students' reasoning about quantities, relationships, and observable aspects of dynamic situations. The first two types of reasoning, *Covariation* and *Variation*, refer to evidence of students' gross coordination of values and gross variation (Thompson & Carlson, 2017). For example, if a student is tasked with sketching a graph of the toy car dynamic situation (Figure 1), they might say, "the distance from the cone decreases at first as the total distance traveled continues to increase, so the graph should start by going down". This response would serve as evidence of Covariation per Johnson et al.'s (2020) framework, as they are attending to the gross change in both attributes. If they only mention the change in one attribute (i.e., "the total distance traveled continues to increase"), their response would serve as evidence of Variation. The third type of reasoning, *Motion*, refers to evidence of students' reasoning about physical motion embedded in dynamic situations (Kerslake, 1977). For example, a student reasoning at a level of Motion might say, "the toy car moves at a constant speed, so the graph should be linear." The fourth type of reasoning, *Iconic*, refers to evidence of students' reasoning about physical features that comprise a dynamic situation (Leinhardt et al., 1990). For example, a student reasoning at a level of Iconic might sketch a graph resembling the path of the toy car, saying, "the toy car's path forms a circle around the cone, so the graph should have a circle too." Johnson et al.'s (2020) framework distinguishes between these four unique ways in which students may conceive of graphs.

Methods

We report on a qualitative case study (Stake, 2005) of a college algebra student, Jamie (pseudonym), who participated in a series of task-based interviews (Goldin, 2000). This study is part of the first author's dissertation, a collective case study conducted as part of a larger National Science Foundation funded project aimed at promoting students' mathematical reasoning by engaging with tasks called "techtivities." Housed in the Desmos Classroom (www.desmos.com), techtivities are interactive tasks based on video animations designed to engage students in quantitative reasoning (Johnson et al., 2020).

The Toy Car Techtivity

In the Toy Car techtivity, students watch an animation of a toy car traveling at a constant speed around a cone (Figure 1). They are asked to consider two attributes: the toy car's total distance traveled and distance from the cone. Next, students are to represent changes in these attributes by dragging "Axis Sliders" along the axes of a Cartesian plane as the animation plays. Students are to sketch a graph relating the two attributes before watching an animation of a computer-generated graph that shows the accurate graphical representation of the relationship.

Data Collection and Analysis

We used purposeful sampling (Gliner et al., 2017) to select Jamie (pseudonym), a college algebra student interested in participating in our study after seeing an announcement posted by her instructor on our behalf. Drawing on teaching experiment methodology (Steffe & Thompson, 2000), the teacher-researcher (TR) conducted exploratory teaching in a series of online task-based interviews (Goldin, 2000), in which Jamie completed four different techtivities. The TR used semi-structured protocols during the interviews.

We analyzed each interview using Wolcott's (1994) description, analysis, and interpretation framework. In the description phase, we documented everything observed during the interviews. In the analysis phase, we inferred Jamie's reasoning based on her actions and utterances. Finally, in the interpretation phase, we connected our findings to related literature, including frameworks from Lee et al. (2020) and Johnson et al. (2020).

Results

Our analysis of Jamie's third interview, during which she completed the Toy Car techtivity, revealed that she demonstrated evidence of multiple conceptions of coordinate systems (Lee et al., 2020) and forms of graph reasoning (Johnson et al., 2020). Through sketches and slider movements, Jamie displayed these constructs in a way that was agentic and meaningful to her.

Conceiving of Attributes as Quantities

Jamie's responses and drawings at the beginning of the techtivity serve as evidence of her conceiving of the toy car's total distance traveled and distance from the cone as quantities (Thompson, 1994, 2011). When the TR asked Jamie to describe the toy car's distance traveled, Jamie traced the entire path of the car with a sketch tool and said, "As the car's moving, he's getting further in distance." When the TR asked Jamie to describe the toy car's distance from the cone, Jamie drew a straight line between the toy car and the cone and said, "The distance is that, but if he's like in this area [puts cursor near the cone], the distance from the cone is way shorter."

Representing Different Aspects of the Situation with Sliders

Jamie dragged the horizontal Axis Slider (intended to represent Distance Traveled) from left to right along the x-axis as the toy car traveled. She followed this up by saying, "I just kept moving my slider until it basically stopped moving, and I tried to go at the same speed." This response, along with her previous description of distance traveled ("As the car's moving, he's getting further in distance"), demonstrate evidence of Variation and suggest that Jamie used the slider to represent the change in the toy car's distance traveled using the horizontal slider.

Next, Jamie dragged the vertical Axis Slider (intended to represent Distance from Cone) upwards along the y-axis as the car traveled towards the cone, downwards along the y-axis as the car traveled around the right side of the cone, and then upwards again as the car traveled around the left side of the cone and continued away from the cone (Figure 2). She went on to describe her slider movements:

> So I assumed since [the slider] was in the middle of the graph, that it would want me to go back down when it did that little loop. So that's what made me move it. Like that's what made me move [the slider] down and then go back up 'cause it does do a little circle.

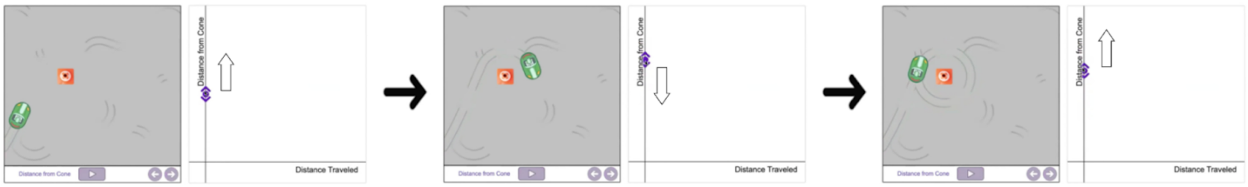

Figure 2. Jamie's Movements with the Vertical Axis Slider (Distance from Cone).

Jamie's movements with the vertical Axis Slider suggest that she was representing spatial aspects of the dynamic situation. She appeared to use the slider to match the toy car's vertical displacement from the bottom of the screen. Jamie's response shows evidence of Motion, as she is describing how her movements with the slider were meant to match the position of the toy car.

Demonstrating Different Conceptions

Jamie proceeded to sketch a graph on the Cartesian plane (Figure 3). The TR asked Jamie to describe her sketch, and she proceeded to talk about different sections of her graph.

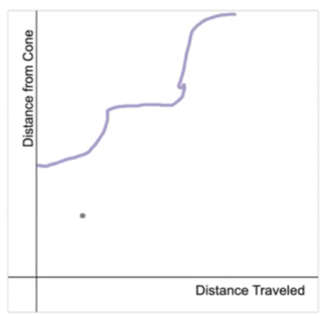

Figure 3. Jamie's Graph Sketch.

Jamie: Yeah I feel like it might be going up, and then possibly, like I don't know if a graph could do this, but plateau out and then like all the way up.

TR: And can you talk more about that plateau?

Jamie: The plateau is when it's in the circle when it's staying at that one point, but the distance is still moving. So that's why I drew that there. And then after once it gets out of that circle, the distance from the cone goes back up along with the distance traveled.

Jamie's responses highlight multiple forms of graph reasoning. Her first comment, "it might be going up," demonstrates evidence of Motion, as she is attending to the movement of the car in space. Her second comment, "the distance from the cone goes back up along with the distance traveled," demonstrates evidence of Covariation, as she is describing the relationship between attributes.

Jamie proceeded to the last part of the techtivity and watched an animation of the computer-generated graph. When the TR asked Jamie for her initial reaction to the graph, she talked about the difference in how the computer treated the attribute on the y-axis.

With their graph, they moved that dot down and then they go back up. I didn't think to move it down because when I did my graph, when I did the little move thingies, I more followed the car. Like the car's going up, so I'm going up. And then when the car came back around, I stayed and then I went back up.

Jamie's responses indicate that she situated her original graph (Figure 3) in a coordinate system containing both quantitative and spatial components. The left-most section of Jamie's graph represents the path of the car (i.e., "it might be going up"), indicating a spatial coordinate system. However, the middle ("plateau") and right-most sections represent the gross change in attributes (i.e., "the distance from the cone goes back up along with the distance traveled"), indicating a quantitative coordinate system.

Discussion/Conclusion

We explored the nature of a student's conceptions of coordinate systems (Lee et al., 2020) and graph reasoning (Johnson et al., 2020) within a digital graphing task involving a dynamic situation. Throughout the course of one interview, Jamie demonstrated evidence of both kinds of conceptions of coordinate systems (spatial and quantitative) as well as different forms of graph reasoning (Motion, Variation, and Covariation). Notably, she drew on these different ways of reasoning in ways that supported her goal of graphically representing the dynamic situation. Jamie's case speaks to the complexity of students' conceptions of coordinate systems and graph reasoning.

Our study supports the notion that students' conceptions of coordinate systems (Lee et al., 2020) and graph reasoning (Johnson et al., 2020) are interconnected (Johnson et al., 2024; Paoletti et al., 2022). On the screen with the Axis Sliders, Jamie demonstrated Variation when representing the change in attribute, and Motion when representing spatial aspects of the dynamic situation. While sketching a graph, Jamie demonstrated Covariation in a quantitative coordinate system and Motion in a spatial coordinate system. In other words, Jamie's conception of a quantitative coordinate system related to her reasoning about changing attributes and their relationship, while her conception of a spatial coordinate system related to her reasoning about the physical motion embedded in the dynamic situation.

Our analysis is ongoing. Jamie is one of three students who the TR has conducted multiple interviews with using multiple different techtivities. We are continuing to employ exploratory teaching (Steffe & Thompson, 2000) and use Wolcott's (1994) framework to learn about how students' conceptions of coordinate systems (Lee et al., 2020) and their graph reasoning (Johnson et al., 2020) are related.

Acknowledgments

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