Exploring the Concept Image and Concept Definition of Periodic Functions

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This paper presents the results of a study where a student's concept image and concept definition of periodic functions were explored. The findings suggest that a student thinks about periodic functions as visual repetition of patterns within a function's graph. He also analyzes output values, particularly at distinguishable points such as maxima, to determine whether a function is periodic. His concept image and concept definition include the idea that a function's period represents the length between any two points where the output values are the same and the pattern repeats throughout the domain.

Keywords: Periodic Function, Concept Image, Concept Definition

Introduction and Background

The periodic function is an important mathematical concept that plays a central role in understanding various topics in mathematics, physics, and engineering (Jia et al., 2011; Mays & Loverude, 2019). Some researchers have mentioned the concept of periodic functions through the lens of students' understanding of trigonometry and focused on periodicity as a property of trigonometric functions (Weber, 2005; Ortega, 2017; Brown & Rice, 2011). Dreyfus and Eisenberg (1980) noticed that although most students hold an intuitive understanding of the periodicity of a function, applying the properties of a periodic function in different contexts is often difficult for them. Although the literature on students' thinking of periodic functions is rare, I found a limited amount of work that discusses students' understanding of the concept of periodicity. The analysis of Buendía and Cordero's (2005) study revealed that predicting future movements based on the movements of the graph of a function is a characteristic that students may use to make meanings for the notion of periodicity. Shama (1998) investigated ways of understanding periodicity. The findings suggest that (1) the students consider periodicity a dynamic collection of elements (time dependency, motion, etc.), (2) the students think about periodic phenomena as a whole unified structure (the segment bounded between two points in one-dimensional space is considered to be a period as whole). Although both studies (Shama, 1998; Buendia & Cordero, 2005) primarily focused on students' perception of periodicity, they did not delve much into the role of periodicity in the concept of periodic function in mathematics. This study aims to contribute to the limited literature on the students' thinking of a periodic function.

Therefore, this paper's motivation is rooted in the fact that the mathematics and STEM education research community will benefit from learning the students' conceptual understanding of a periodic function. The results of this study provide opportunities for researchers to explore the role of a periodic function in various mathematical topics and use the ways of thinking to teach students. In this paper, I explore how an undergraduate engineering student describes his thinking of a periodic function while he works through a series of tasks.

Theoretical Perspective

The constructs *concept image* and *concept definition* were first introduced in a PME article (Vinner and Herschowitz, 1980). Then Tall & Vinner (1981) published the first journal article on these constructs and applied them in reference to limits and continuity. This research uses *concept image* and *concept definition* to frame a student's thinking about periodic functions. These constructs provide researchers with viable ways to interpret the data and categorize students' mental pictures based on what they say, do, draw, point at, etc.

Tall & Vinner (1981) introduced the concept image as follows: "*We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p. 151).* The concept definition was mentioned as follows: *"We shall regard the concept definition to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his (evoked) concept image. Whether the concept definition is given to him or constructed by himself, he may vary it from time to time. In this way a personal concept definition can differ from a formal concept definition, the latter being a concept definition which is accepted by the mathematical community at large" (p. 152).*

We have adopted this framework to analyze our data. We use the construct *concept image* as a whole structure that includes evoked mental snapshots of the student we interviewed while he worked through a series of tasks about periodic functions. We also use the term *concept definition* to describe the evoked personal definition he developed from his intuition after exposure to the tasks we presented. We address the following research question in this study by adapting the concept image and concept definition as our theoretical background-
What concept image and concept definition of periodic functions may a student hold while he works through a series of tasks?

Methods

Data Collection

I conducted a semi-structured clinical interview (Clement, 2000) with one participant, Fred (pseudonym). He was enrolled in an engineering major at a large southwestern university. The only requirement to participate was to have completed at least the first level of the Calculus course. Thus, Fred was familiar with the concept of functions, and upon asking, he mentioned that he associates periodic functions with trigonometry. However, periodic function concepts were never discussed in his Calculus classes. Fred participated in two interviews (each about an hour long) over Zoom. The sessions were video recorded via Zoom. The interviewer shared a screen with Fred with the tasks for each session. Blank spaces between the questions allowed him to annotate or write to capture his ideas and diagrams. He worked on a total of 12 tasks during the interviews. The tasks involved some open-ended questions such as: Have you heard of the concept of 'periodic function?' If yes, can you please say about your conception of periodic function? Some tasks presented graphical and imaginary situations to the participant, and interview questions were designed to learn what thinking process students use and how they use it in specific given scenarios.

Data Analysis

For analysis, I utilized in-vivo coding (Saldaña, 2021). A researcher uses in-vivo coding when s/he focuses on the words from data as a first step to code. I first transcribed using Adobe Premier Pro and then manually corrected auto-transcription errors. I read through the data and named codes based on words and phrases. I labeled the data into discrete parts based on the codes I created. Next, I compared similar events in the data and collated excerpts that were labeled with a particular code. Then, I drew connections between the codes with the goal of finding how the codes can be grouped into categories. These steps of analysis aimed to open new theoretical possibilities. I used the terms concept image and concept definition constructs to frame the thinking of periodic functions students inhibited during the interviews by thinking out loud, drawing, gestures, pointing out, etc.

Results

In this section, I highlight the various ideas of periodic functions in Fred's concept image and his choice of a personal concept definition that has evolved while he worked through Tasks 1-7.

Periodicity as Repeating a Pattern

Fred was never exposed to the formal definition of a periodic function. Tasks 1 and 2 explored the student's current conception of periodic functions, followed by an example. In response, Fred described periodic functions as repeatable functions. To elaborate on Fred's description of a 'repeatable function,' he used the example of the sine function-

> "So, I mean, like, there might be some function like probably $\sin(x)$. Right. So, it would go like it would be a periodic function starting from zero that would go on to one and then come down again and then come to zero again. Right. So, if that's a period, it would be the same. The same. How do I say this? The same pathway of that function would be repeated after the period itself, like the period of the function".

I note that Fred was imagining the graph of a sine function within the interval of 0 to 2π. He then imagined the same pattern of the graph being repeated onwards. Here, Fred was thinking about the unchanged pattern of the graph within an interval as the period of the function.

Focus on Output Values

Task 3 presented a graph of a function that represents a recurrence pattern but does not return the same output value for any period. Fred acknowledged that the pattern is repeating but also mentioned that the graph does not represent a periodic function. He noticed that although the pattern is similar, the function does not have the same output value for any period. To validate his reasoning, shown in Figure 1, Fred chose segments of patterns within the graph. First, he separated intervals where the pattern repeats, and then he chose half segments within those intervals. He chose imaginary points where he assigned half segments of the pattern within the first two intervals, and then he chose imaginary corresponding output values for the input values. Then, he drew horizontal and vertical lines to show how the output value of the function does not return for a period.

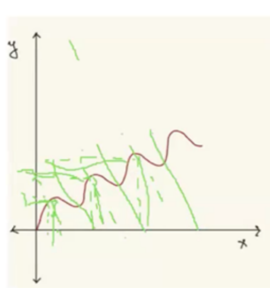

Figure 1. Fred's scratch work shows how the function does not return the same output values for any period.

Task 4 presented three apparent graphical examples of periodic functions, and one graph of a non-periodic function and the interviewer asked Fred to determine which of the following graphs represents periodic functions. This task aimed to determine whether a student demonstrates a consistent conception of periodic functions in relation to his/her previous responses and to learn what thinking process students use and how they use it to determine whether a function is periodic from graphs. Looking at the graphs, Fred quickly determined the three graphs representing periodic functions. The interviewer then asked him to explain his rationale for choosing each graph as a periodic or non-periodic function. Fred chose the word cycle to refer to one complete graph pattern within a period. To justify the first graph of the choices as a periodic function, his responses are as follows-

Fred: So suppose we are looking at this point x equals to something between -2π to $-\frac{3}{2\pi}$, okay? And if we take this one here (Fred points to another input value on the graph for which the output is the same as the point he initially chose) it would give the exact y value. It's the same y value for that point. That is why I would say this is a periodic function. And the cycle repeats itself. And it's exactly the same over here on the left side as on the right side. And it also, like the same interval, same time values. If you start from here, then after that, it would give you the same.

Interviewer: so, when you say, start from here, what do you mean? …

Fred: ….So, start over here, I mean, when the cycle starts here, (pointing to a value in the graph), we mentioned like…

Interviewer: -2π?

Fred: …Yeah. When the -2π starts and after a small interval…..probably like π over this (pointing to another point where the repetition occurs). If we consider this point, this would give the exact same y value as if we started here and took the same interval, like 0.2π. … It would give the same exact y value. So that's the reason I think it's periodic.

Interviewer: Okay…then like what is the period? How would you determine the period of this function?

Fred: Yeah. So, the same thing as I said before. So, whenever we have a function, and it starts, the cycle starts repeating itself, and we can see a trend. Right? So, that's how we divide it (he made segments of repetitive patterns in the graph). So, I can see that at x equals to 0, there, the cycle starts repeating itself again from here to here (pointing to the pattern within a period in). And it's the same exact function that starts from here again.

Figure 2 represents Fred's annotation in relation to the excerpt above.

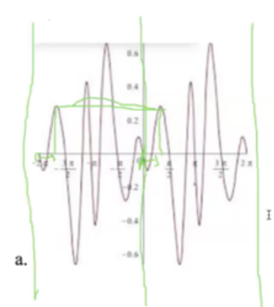

Figure 2. Fred shows repetitive patterns in the graph with the repetition of output values.

To determine whether a function is periodic, Fred initially identified a pattern of the graph that repeats, and he referred to the pattern within a period as a cycle. Then, in support of his argument, Fred chose two points where the output values are the same for two different input values. He thinks about periodic functions as functions when the same pattern repeats after a period, and the output values repeat for different input values after each period. Figure 2 shows Fred's way of separating periods of a function and identifying the same output values for different input values as the pattern repeats. Fred drew a green horizontal line from one point to another to show an interval between two same output values. He also identified this length of the horizontal green line as the period of the function. Fred maintained similar reasoning to identify other periodic functions in task 4.

Fred's Concept Definition

Task 5 presented five statements that describe periodic functions. Fred used the ideas of repeating a pattern and finding the same output values while he comprehended those statements. I designed task 5 anticipating that students would use these statements to organize the cognitive structures they would have developed about periodic functions.

Task 6 asked Fred to reflect on the tasks he had worked through thus far and to define a periodic function. Initially, Fred mentioned that a periodic function is like a cycle that repeats itself, and for a specific interval after the cycle starts, it would give the same output value. He drew Figure 3 to elaborate on his definition of a periodic function.

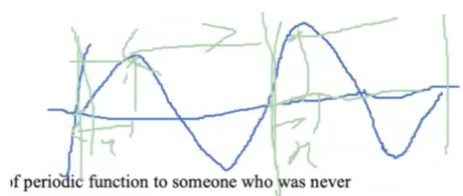

Figure 3. Fred shows a periodic function with two repeating cycles.

Here, Fred drew the graph of a function with two repeating patterns. He wrote two x's to show the starting points of each cycle. He also highlighted the length between two maximum points in the graph to show the period of the function. The interviewer asked Fred to pretend to explain the concept of periodic functions to someone who had never been exposed to it and how Fred could tell whether a function is periodic or not. Fred mentioned that he imagines a periodic function as a uniform repetition of patterns. In his words-

> "These two parts (referencing two repeating patterns in Figure 3) are uniform here. That's (uniformity) one of the criteria is that we can define it as a periodic function. And then again, we have the fact that a specific interval, for example, this (choosing $(0,0)$ point) is a starting point of this period for a specific interval. This length, (a length Fred chose named x in the graph) probably x, will give us the same y (output) values here from x (pointing at the starting point of the second repeating pattern). Here, it's going to be exactly same y values here. So, two things to keep in mind are uniformity and the cyclical nature of the function. And the other thing is that whenever we take a small interval after the periodic function starts, it would give us the same output on both intervals".

Fred summarized his concept definition again when the interviewer asked how he could tell whether a function is periodic. Fred defined periodic functions as functions that maintain a uniform repetition of patterns throughout the domain after certain intervals, and the functions have a cyclical nature. He also included the idea of the same output values in his definition, as he mentioned that for a function to be periodic, we can choose any length of intervals from the starting points of each period, and the output values for the corresponding input values would remain the same. As we see in Figure 3, Fred noted two lengths of intervals named x's from the starting points of each segment of repeating patterns. Fred mentioned that the output values for the corresponding input values in both intervals would be the same.

The Distance Between *Turning* Points is the Period

Fred's primary criterion for determining a periodic function is to look for a pattern that repeats throughout the function's domain. However, to locate the period of the function from the graph, Fred relied on choosing a distance between two maximum points instead of choosing the distance between the starting and end points of one pattern that repeats itself. The interviewer asked his reasoning for choosing these points, and he replied that these are turning points in the graph. Fred used the term turning points to note the visually easier points (shown in Figure 3) so that he could measure the length in between without calculations.

In task 7, Fred encountered three imaginary periodic situations. The interviewer asked him to represent the context graphically and determine the function's period. The first context given was- *when the heart contracts, blood pressure in the arteries rises rapidly to a peak (systolic blood pressure) and then falls off quickly to a minimum (diastolic blood pressure). Blood pressure is a periodic function of time.* Fred chose some imaginary measurement of the blood pressure. The blood pressure would vary between 65 to 115 units of measurement, and at the initial moment, the blood pressure would be 90 units of measurement. A part of Fred's reasoning for graphing the situation and finding the period is mentioned here-

Fred: I am assuming that we're starting in the middle….90 blood pressure. I'm not sure about the unit…..so this goes up from there, the maximum which is 115. And it reaches a maximum of 115. It goes down again to the minimum. To 65. Then, when it hits the minimum, it starts going up again. And it hits 115 again. It starts coming down again, and the *cycle* keeps repeating itself. So, here we are considering that we're starting from the 90 and we're just stopping at 90 here as well. Let's assume that this is two seconds that it took for one *cycle* to be completed. So, two seconds would be the period for this function.

Interviewer: So, where does the *cycle* start, and where does it end? Tell me this, using the context.

Fred: So, for this case, …..we could just make it easier for us to say,….whenever we reached the maximum pressure in the heart, like blood pressure. The next time, we reached the maximum blood pressure again. We can say that it is one *cycle*.

Interviewer: Can you use a horizontal line to show me one *cycle*?

Fred: Yeah, this would be one cycle.

Interviewer: So, one maximum to another maximum?

Fred: Yeah.

Interviewer: Okay, so that's one cycle. And what is the period?

Fred: It would be the same. The period here is two seconds. That's what we assumed. Suppose this is 0.5, and here, it's 1, it's 1.5, this is 2, and this is 2.5. So, 2.5-0.5. That would give us 2.

I note that Fred used the words cycle and period interchangeably when he was thinking about the cycle length as the period of the function. To determine the period of the function from a graph, Fred relied on the easily distinguishable points on the graph. While working on task 6, Fred named his choice of maximum points (shown in Figure 3) as turning points when he was using a graph of a periodic function without context to determine the period of the function. His reasoning for choosing turning points to determine the period of the function was sustained in task 7. However, this time, Fred was graphing a real-life situation, and he assigned numerical output values. Fred chose the distance between two maximum points of the function to determine the period. The contrast between tasks 6 and 7 is that in task 6, Fred chose a cycle from the starting to end point of a pattern that repeats, but in task 7, Fred mentioned the pattern between two maximum points as one cycle. When the interviewer asked Fred to determine the period of the function, he assigned numerical input values. He determined the distance between one maximum point and another as the period of the function.

Discussion

In this section, I summarize the important findings and discussion points based on the results. I highlight the key ideas of periodic functions in Fred's concept image and distinguish his personal concept definition of a periodic function from a formal definition.

Fred's concept image of periodic functions centers on the idea of a repeating, unchanged graph pattern. He refers to this as the same pathway of the function, emphasizing the consistent repetition of the graph's structure. To determine whether a function is periodic and to identify its period, Fred primarily focuses on the function's output values. He prefers choosing clearly distinguishable points on the graph, such as maximum points, as reference points. According to Fred, selecting these prominent points makes it easier to visualize the period of the function without needing to assign specific numerical values. Moreover, Fred's concept image includes a flexible understanding of the period as the distance between any two points where, for different input values, the output values return to the same value. He perceives the pattern between these two points as repeating consistently for subsequent intervals throughout the domain.

Dreyfus and Eisenberg (1980) formally defined a periodic function as $f(x)$ being periodic if there exists a positive number, p, such that $f(x+p) = f(x)$ for all x in the domain of f. In contrast, Fred's concept definition focuses on the idea of repeating patterns that remain consistent throughout the domain and the recurrence of the same output values after each period. Since Fred had not encountered the formal definition of periodic functions, it is reasonable that his concept definition did not include any mathematical notation. Instead, Fred relied on an evolving, descriptive understanding of a periodic function and how to recognize periodicity based on patterns without formal mathematical expressions.

Fred's understanding of periodic functions reveals an intuitive grasp of the concept, despite his lack of exposure to formal mathematical definitions. His focus on visual patterns and the repetition of output values allows him to construct a personal definition that aligns with certain aspects of the formal definition. By examining Fred's understanding of periodic functions, the study highlights how learners can meaningfully engage with mathematical concepts through visual and experiential reasoning, even in the absence of formal definitions. By understanding how individuals like Fred perceive mathematical concepts, educators can tailor their teaching strategies to help students connect intuitive understandings with formal mathematical language. Overall, Fred's case emphasizes the value of concept image and concept definition as a foundation for developing a deeper comprehension of mathematical ideas.

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