What Makes Content Knowledge *Specialized*: A Case Study of Graduate Teaching Assistants' Mathematical Knowledge for Teaching

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As the push for active learning pedagogies in undergraduate mathematics continues, research into the teaching practices of graduate student instructors (GSIs) has expanded significantly. This paper presents findings from a multiple-case, descriptive case study of the demonstrated mathematical knowledge for teaching (MKT) of GSIs within the context of active learning Precalculus lessons. This study aims to bridge the theoretical and practical distinctions between the MKT knowledge domains as they are used by GSIs in their teaching practice. Practical implications for the professional development offered to GSIs, as well as theoretical implications for studying the MKT construct in post-secondary settings, are discussed.

Keywords: Mathematical Knowledge for Teaching, Graduate Student Instructors, Precalculus

Several decades of research comparing active learning to traditional lectures in STEM classrooms has shown significant improvements in student learning outcomes in classrooms where active learning pedagogies are used (Freeman et al., 2014). As a result, researchers and educators have increasingly called for the implementation of active learning in undergraduate mathematics courses (Akin et al., 2023). Active learning is an umbrella term for classrooms where students engage in mathematical activities that promote higher-order thinking, such as group problem-solving and investigations (Conference Board of Mathematical Sciences, 2016). Active learning is often viewed as more difficult to implement than traditional lecture because the instructor must give over some control to allow their students to engage meaningfully with the mathematics. When student thinking is centered in the classroom, the instructor must balance opportunities for inquiry with structured guidance toward correct mathematical understandings (Akin et al., 2023). At Ph.D. granting universities in the United States, it is common for mathematics Ph.D. students to receive funding for their studies by teaching introductory undergraduate mathematics courses, such as Precalculus (Beisiegel et al., 2019). As a result, many of the introductory mathematics courses at Ph.D. granting institutions are taught solely by graduate student instructors (GSIs) (Deshler et al., 2015). It can be daunting for novice instructors, such as GSIs, to implement active learning pedagogies in their courses without clear guidance and support (Beisiegel et al., 2019). As the push for active learning in undergraduate mathematics increases, there is a concurrent push to understand the knowledge that GSIs need to implement active learning pedagogies in order to develop effective teaching supports (Akin et al., 2023). The purpose of this study is to examine the knowledge that GSIs use within their teaching practice and identify the purpose(s) within their practice that their knowledge serves.

Theoretical Framing

It is commonly accepted that instructors need content knowledge to teach effectively, but that content knowledge alone is insufficient (Ball et al., 2008). From analyzing teachers' practice, Ball et al. (2008) hypothesized that six distinct but related domains of knowledge "made up" MKT. Following Ball et al.'s (2008) seminal piece defining MKT, research about teacher knowledge has revolved around refining the theoretical definitions and creating clear delineations between the domains (e.g., Copur-Gencturk et al., 2019). Copur-Gencturk and her colleagues (2019; 2022) conducted several quantitative studies to disentangle the subdomains of knowledge in Ball et al.'s (2008) framework. Copur-Gencturk and Tolar (2022) refined their definition of MKT to include "teachers' understanding of the mathematics, their pedagogical content knowledge (i.e., knowledge of students' mathematical thinking, their instructional practices, and the representations they use), and their ability to notice content-related issues in the moment of mathematics teaching" (p. 2). Their findings concluded that, statistically, content knowledge, pedagogical content knowledge, and content-specific noticing skills are distinct elements of MKT among their participants (See Figure 1). Copur-Gencturk et al.'s (2019; 2022) work is important in pushing forward the field's understanding of teacher knowledge from a theoretical perspective. Methodologically, however, these studies employed large-scale written assessments and quantitative analyses rather than using Ball et al.'s methodology of qualitatively examining teachers' practices, which may limit the applicability of the MKT framework for informing efforts to develop teaching supports for GSIs.

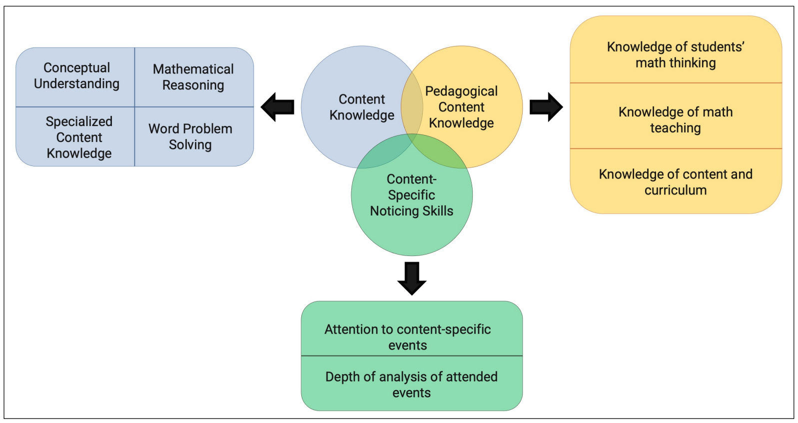

Figure 1. Mathematical Knowledge for Teaching as Conceptualized by Copur-Gencturk and Tolar (2022)

When extending ideas of MKT to post-secondary contexts, the theoretical distinctions between knowledge domains become more unclear (e.g., Howell et al., 2016; Speer & King, 2014; Speer et al., 2015). A question raised in Speer et al.'s (2014; 2015) work refers to how common content knowledge (CCK) and specialized content knowledge (SCK) are currently defined. Theoretically and practically, the concern is that knowledge considered as common for mathematics Ph.D. students versus those with only a bachelor's is most likely different in nontrivial ways. While assessing the mathematical validity of a student's response is unanimously accepted as SCK for an elementary teacher, this ability and knowledge may be considered common within the work of a Ph.D. student. For theoretical purposes, Copur-Gencturk and Tolar (2022) acknowledge that both domains overlap and, therefore, their definition for SCK is content knowledge "situated within the task of teaching, whereas the other components of [CCK] are not dependent on the teaching context" (p. 4). In other words, SCK can be operationalized as the mathematical knowledge that is used for purposes such as assessing students' mathematical reasoning, evaluating students' (potentially unconventional) solution strategies, and providing explanations of concepts to students. The study presented here is positioned to address concerns such as those raised by Speer et al. (2014; 2015) and to add to the current theoretical understanding of MKT by incorporating distinctions that come from the examination of GSIs' teaching practices. Specifically, this study is positioned to (1) identify the domain(s) of knowledge that GSIs use when using active learning pedagogies and (2) identify the purpose(s) for which their knowledge is employed when using active learning pedagogies.

Methods

This study followed a descriptive, multiple-case study design (Baxter & Jack, 2008) of mathematics GSIs for Precalculus at a large research institution in the Southeastern United States during the Fall of 2023. The study design is foremost descriptive in nature because the goal was to examine the real teaching practices of the GSIs in context and come to a more nuanced understanding of how their MKT was utilized in practice. The mathematics department at this university had recently undertaken several initiatives to improve student learning outcomes in introductory undergraduate courses. Such initiatives included creating and facilitating GSI professional development (PD) for Precalculus to support the implementation of active learning pedagogies. Participants in this study were three current Ph.D. students in the mathematics department who were GSIs for Precalculus. All of the participants in this study had completed at least two years of their programs and had successfully completed their qualifying exams. This paper presents the findings from one participant, Leona. At the time of data collection, Leona was a third-year Ph.D. candidate in Applied Mathematics. She had taught Precalculus in the year prior to data collection during the first semester when the new Precalculus units and PD were integrated into the curriculum. Leona was chosen as the first case because she was the only participant with previous experience as an instructor of record for this course.

Data Collection

MKT is employed in all aspects of a teacher's work; therefore, collecting multiple data sources is necessary to create a holistic view of the components of MKT as they are being used together (Meijer et al., 2002; Park & Oliver, 2008). To this end, data for this study were collected from two semi-structured interviews at the beginning and end of the semester, five PD meetings, and four classroom observations. The classroom data were split between two lecture classes and two active learning lessons focused on exponential functions. Data collected during the active learning lessons were the primary source used to address the following research question: How are the domains of Mathematical Knowledge for Teaching demonstrated in the teaching practices of graduate student instructors during active learning Precalculus lessons focused on exponential functions?

Data Analysis

All data were audio- and video-recorded before being transcribed using Temi. The researcher edited transcriptions for clarify and accuracy. The transcript data from the active learning lessons were chunked into "exchanges," defined by the GSIs' interactions with one or more students. Each exchange captured one distinct interaction between the GSI and their students. Exchanges would begin and/or end when the GSI stopped interacting with one group of students and began interacting with another. Each exchange served as the unit of analysis for data coding procedures. Following the chunking of the lesson transcripts, the first data analysis phase consisted of re-reading the lesson transcripts while re-watching the instruction videos. During this review, I wrote detailed descriptive memos about the exchanges. These memos focused on summarizing the interactions between the GSI and their students. The next data analysis step was assigning knowledge domain codes to each exchange. MKT knowledge domain codes were assigned based on whether the GSI demonstrated the domain of knowledge during the exchange. The codebook for the domains of MKT was created through the incorporation of Copur-Gencturk and Tolar's (2022) Content-Specific Expertise for Teaching Mathematics framework with the content-specific definitions for exponential functions from the Exponential Growth Learning Trajectory (Ellis et al., 2016). The second round of data coding addressed the link between domains of knowledge and teaching practices. Units of analysis continued to be the exchanges. The teaching purpose codes were developed through a literature-informed, inductive method. In other words, the literature on teacher knowledge highlighted several purposes for which teacher knowledge could be used. These purposes included selecting and sequencing classroom tasks, assessing student understanding, and informing in-the-moment classroom decisions (e.g., Park & Oliver, 2008). While these teaching purpose codes were not defined in an a priori codebook, the code names and definitions were developed using a combination of language from the literature and the participant's words and actions.

Findings

I found that Leona most commonly demonstrated a combination of content knowledge and knowledge of the curriculum for the purpose of building mathematical connections between student thinking and key learning goals. Specifically, she demonstrated knowledge of exponential functions and the lesson's learning goals and task sequence when responding to students' questions and providing guided explanations (see Figure 2). Within these exchanges, Leona appeared responsive to her students' thinking and flexible in her responses or explanations. The curriculum was designed to elicit specific ways of reasoning about exponential functions. Teacher notes were embedded within the Desmos lessons, which were discussed with the GSIs during the PD meetings. The teacher notes integrated examples of common student responses with pedagogical suggestions designed to align with the goal of promoting productive covariational reasoning and building on current student understanding. Leona's interactions with her students often parallel the suggested pedagogical moves from the curriculum or suggestions discussed during the PD meetings. While her content knowledge was demonstrated throughout these exchanges, it was especially evident when she would bridge concepts and representations in ways that were not previously discussed with other GSIs or suggested by myself or the course coordinator.

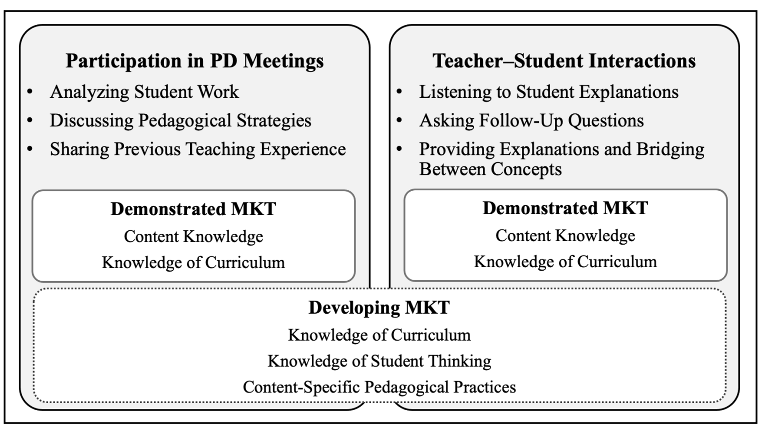

Figure 2. Visualization of Primary Finding for Leona's Case

Knowledge of Curriculum: Learning Goals and Task Sequence

One such student interaction occurred in the first of two active learning lessons, where a student was working on the questions from Figure 3. In these questions, students are first asked to identify two unknown heights (y values), given information about the amount of potion that Alice consumed (x values), and explain how they found one of the height values. Students are then given how Alice's sisters, Kadesha and Susana, found each of the values and asked to compare their own explanations with those given by Kadesha and Susana. In this one-on-one interaction, a student asks for Leona's input on their solution because their method does not match either Kadesha's or Susana's method. The student wants Leona to check her work and help her understand the methods shown in the task.

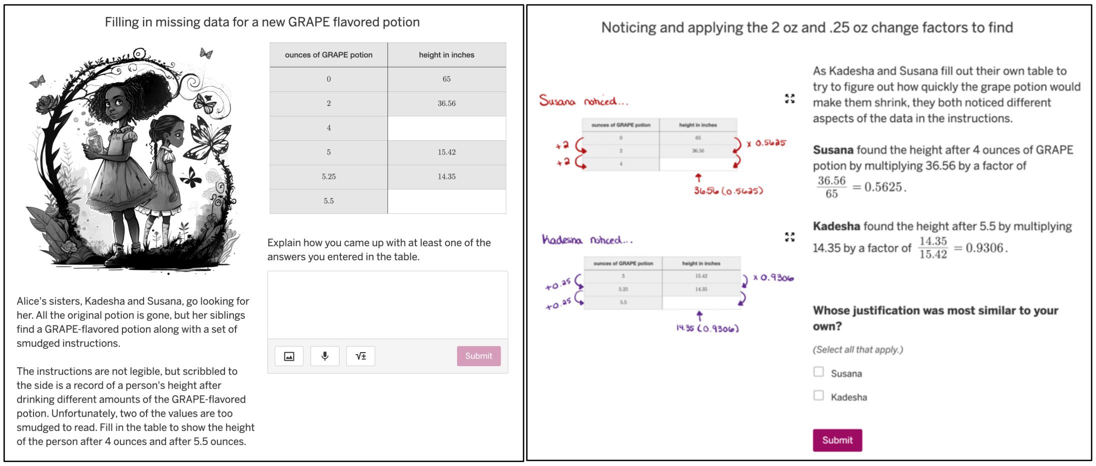

Figure 3. Student Facing Questions in Desmos Lesson

During this interaction, Leona asks the student to explain their reasoning so that she can first understand what the student has done before providing any other explanations or asking other questions. The student solved this task by substituting values from the table into the standard exponential function formula ($y = ab^x$) and solving for the base of the exponential. This lab's learning goal is for students to coordinate successive additive increases in ounces of potion with proportional multiplicative changes in height. The student's solution, while mathematically correct, relies on an algebraic representation of exponentiation, which does not push the student's understanding of the covariational relationship forward in the way that this task was intended to. Leona uses her knowledge of this learning goal and how the tasks are sequenced to inform her decision on how to structure the questions that she asks this student. Specifically, her questioning builds key mathematical connections between the student's solution strategy and those presented in the task from Susana and Kadesha. Additionally, Leona returns to the algebraic representations to help the student make another connection between the standard representation of exponential functions and a representation using the rate found by Susana. This further connection demonstrates Leona's knowledge of the task sequence because she is setting up the foundation of ideas that will become relevant in the second lesson on exponential functions.

Content Knowledge: Exponential Functions

The student in this interaction used the standard exponential function equation to find the unknown height values. Leona demonstrates her content knowledge of exponential functions throughout this interaction. Specifically, her content knowledge is being transformed into specialized content knowledge through her interaction with this student because she first must assess the mathematical validity of the student's solution and then use a series of prompts to draw mathematical connections between the student's thinking and the covariational relationships meant to be explored in this lesson. In the first part of this interaction, Leona is using her content knowledge of exponential functions and general algebraic rules to assess the validity of this student's method. There is a minor error in the student's algebra when solving for the base x in the equation: $36.56 = 65x^2$. Leona notes that the student seems to have reversed the order of operations. In other words, the student applied a square root to the equation prior to dividing by 65.

Following this comment, Leona continues to demonstrate her content knowledge in conjunction with her knowledge of learning goals by guiding the student to draw important connections between her algebraic representation and the strategies presented by Susana and Kadesha in the task prompt. Leona draws on her understanding of exponential covariation defined by constant multiplicative proportions for successive n-unit additive changes and its connection with the algebraic representation of exponential functions throughout this interaction. In the following excerpt from the lesson transcript, Leona guides the student to make the connection regarding constant multiplicative proportions by highlighting how Susana's method finds a 2-ounce scale factor and the student's method finds a 1-ounce scale factor. Additionally, they verify that both of these scale factors will arrive at the same answer for the unknown height at four ounces (see Figure 4 for Leona's written work accompanying her verbal explanation).

Leona: But what is x changing by for you personally? For [Susana,] she did 36.56 over 65, right? Every time she does the multiplication…her original answer is 65 times this [ratio], right? This is what happens when x is increased by two. So, she starts at zero [ounces], then x equals x plus two, that's gonna be 65 times (36.56 over 65) times [36.56 over 65] again. But this ratio [36.56 over 65] is for how many?

Student: Two?

Leona: Yeah, so [Susana's ratio] is for two ounces.

Student: Okay. Oh, so there oh, oh, oh, oh, okay. Okay. Because [Susana's] is two. So then [multiplying again] would make it four. This would get you to four [ounces]

Leona: Which would get you your answer. But you are multiplying 65 by 0.75. How many times do you have to multiply by 65 to get [Susana's] same answer?

Student: Four.

Leona: Right? So, you have to do this [multiplication by 0.75] a total of four times, right? So that means every [multiplication of 0.75] is for how much?

Student: One [ounce].

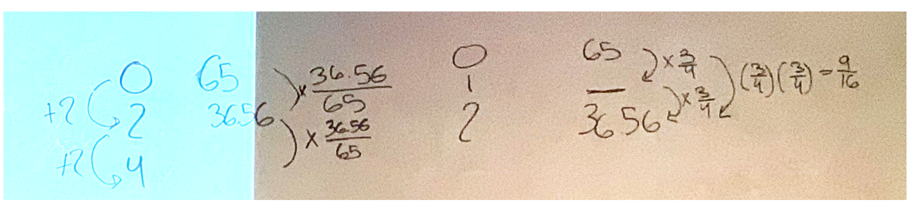

Figure 4. Leona's Written Annotations from Desmos Slides

Discussion

The overview of this finding is that Leona's content knowledge of exponential functions is used in conjunction with knowledge of the curriculum for the purpose of scaffolding interactions with students and building mathematical connections between student reasoning and key learning goals of the lesson. Copur-Gencturk and Tolar (2022) define specialized content knowledge as content knowledge used in teaching practices. In other words, content knowledge that is used for a pedagogical purpose is considered *specialized,* whereas general content knowledge refers to any content knowledge that does not rely on a teaching context. In the example above, Leona's use of content knowledge to make in-the-moment teaching decisions changes her general content knowledge into specialized content knowledge because she employs it for a clear pedagogical *and* mathematical purpose. It is within the moments of teaching and interacting with her students that Leona uses her content knowledge, as well as knowledge of the curriculum, to inform her responses to students and how she builds key mathematical connections with students.

An important contextual piece to this finding is that the GSIs are provided with PD accompanying each active learning lesson. In these PD meetings, the GSIs are introduced to the lesson's learning goals and how the tasks are set up. They are given opportunities to work through the lessons themselves and discuss areas where they think students might need additional support. Lastly, they are introduced to common strategies that students may use and guided through collaborative brainstorming on ways to respond to students that align with the learning goals for the lessons. Leona actively contributed during these PD meetings by sharing her experiences with students from the prior semester and her strategies for implementing the active learning lessons. This is important because knowledge of the curriculum and content knowledge specific to these lessons was not something Leona necessarily possessed prior to her teaching. The PD providers specifically targeted this knowledge to support GSIs in their teaching practices. Additionally, the PD meetings provide a space for GSIs to develop knowledge related to teaching, such as specialized content knowledge, when they are participating in acts such as anticipating student thinking. The knowledge gained during these PD meetings and demonstrated in Leona's teaching is important when considering how the field makes theoretical distinctions between knowledge domains, such as content knowledge versus specialized content knowledge. Research focusing solely on the theoretical distinctions between knowledge domains may muddy our understanding of MKT as knowledge that is inherently demonstrated through actions within teaching practices. Specialized content knowledge has a history of being defined by a demonstration of specific teacher actions such as "building or examining alternative representations, providing explanations, and evaluating unconventional student methods" (Hill et al., 2004, p. 16). From my analysis of Leona's teaching practice, I believe that developing specialized content knowledge comes from her use of content knowledge in conjunction with other domains for specific pedagogical and mathematical purposes. In this case, her purpose was to understand her student's reasoning and respond to them in a productive way that guided them toward the mathematical learning goals of the lesson. These actions necessarily require content knowledge so Leona can provide clear explanations and correct mathematical reasoning. This study has practical implications for GSI PD because it offers insights into the knowledge that GSIs draw on and how it is used in their teaching practices. Specialized content knowledge can be developed or created by GSIs when they have been given time and space to think deeply about the mathematical content, learning goals, student thinking, and how these come together in the act of teaching.

This study aimed to understand GSIs' MKT by analyzing their teaching practices. The findings provide a lens into the overlapping and interactive nature of GSIs' MKT. The findings presented here support the implication that content knowledge becomes specialized when it is used in concurrence with other domains of knowledge for pedagogical and mathematical purposes. These findings have limitations related to the theoretical conceptualization of MKT and methods of inferring knowledge from observed practices. Future research should consider these limitations when designing studies. Further research should continue to refine the theoretical underpinnings of MKT and explore practical implications for GSI PD and related areas of active learning pedagogy in undergraduate mathematics education.

References

Akin, V., Braley, E., & Bookman, J. (2023). Modeling active learning in professional development for teaching. *The Journal of Faculty Development*, *37*(3), 28–37.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Beisiegel, M., Gibbons, C., & Rist, A. (2019). Teacher interrupted: How mathematics graduate teacher assistants (don't) learn about teaching. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 97–104).

Conference Board of Mathematical Sciences. (2016). *Active learning in post-secondary mathematics education*. https://www.cbmsweb.org/wp-content/uploads/2016/07/active_learning_statement.pdf

Copur-Gencturk, Y., & Tolar, T. (2022). Mathematics teaching expertise: A study of the dimensionality of content knowledge, pedagogical content knowledge, and content-specific noticing skills. *Teaching and Teacher Education*, *114*, 103696. https://doi.org/10.1016/j.tate.2022.103696

Copur-Gencturk, Y., Tolar, T., Jacobson, E., & Fan, W. (2019). An empirical study of the dimensionality of the mathematical knowledge for teaching construct. *Journal of Teacher Education*, *70*(5), 485–497. https://doi.org/10.1177/0022487118761860

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*(1), 11–30. https://doi.org/10.1086/428763

Howell, H., Lai, Y., Phelps, G., & Croft, A. (2016). *Assessing mathematical knowledge for teaching beyond conventional mathematical knowledge: Do elementary models extend?* Annual Meeting of the American Educational Research Association. https://doi.org/10.13140/RG.2.2.14058.31680

Meijer, P. C., Verloop, N., & Beijaard, D. (2002). Multi-method triangulation in a qualitative study on teachers' practical knowledge: An attempt to increase internal validity. *Quality and Quantity*, *36*(2), 145–167. https://doi.org/10.1023/A:1014984232147

Park, S., & Oliver, J. S. (2008). Revisiting the conceptualisation of Pedagogical Content Knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in Science Education*, *38*(3), 261–284. https://doi.org/10.1007/s11165-007-9049-6

Speer, N., & King, K. (2014). *Examining mathematical knowledge for teaching in secondary and post-secondary contexts*. https://www.researchgate.net/publication/237608849

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, *18*(2), 105–122. https://doi.org/10.1007/s10857-014-9277-4