Covariation-in-Pieces: Covariation as a Coordination Class

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Much of the work in covariation, to date, has focused on "large-scale" themes, such as broad reasoning levels and behaviors. We propose that covariation research could be greatly enhanced by attending to the "fine-grained" cognition happening in students' covariational reasoning. We propose that covariation can be considered as a coordination class, which models knowledge as a complex cognitive structure wherein an emergent system of knowledge resources are activated in response to context and cueing. We explain what this perspective on covariation would look like, and we provide an illustrative vignette to explore the power of this perspective.

Keywords: covariation, covariational reasoning, knowledge in pieces, coordination class

Covariation and covariational reasoning have become major themes in secondary and tertiary mathematics education research (e.g., Carlson et al., 2002; Castillo-Garsow et al., 2013; Johnson, 2022; Moore et al., 2019; Thompson & Carlson, 2017). Consequently, deeply unpacking covariation as a construct, and the reasoning done in covariational situations, is critical for building this area of work. To situate this paper, we note that most of the work done to date in covariation has focused on a relatively large cognitive scale, which tends to operate under the assumption of the stability of knowledge – e.g., broad developmental levels that encompass mental actions and behaviors (Carlson et al., 2002; Thompson & Carlson, 2017). We argue these mental actions and behaviors may involve the coordination of less stable knowledge-in-use that is manifold in structure and context-dependent in emergence. Our previous work has started to add a "fine-grained" perspective on covariation (Rodriguez, Bain, & Towns, 2019), and we have highlighted the value of applying a fine-grained lens to consider students' graphical reasoning, including students associating intuitive ideas with graphical patterns (Rodriguez & Jones, 2024). These fine-grained ideas operate at a scale of analysis that affords consideration of the flexibility of a student's knowledge, as well as their knowledge structure's dependence on context and time.

In this paper, we systematize this emerging work by discussing covariational reasoning through a fine-grained constructivist lens, highlighting the affordances provided by situating covariation within coordination class theory (and knowledge-in-pieces broadly). We claim that the importance of doing so is that it provides more accurate modeling of student thinking and it better accounts for conceptual change in developing expertise in covariational reasoning.

Brief Literature Review on Covariation

Covariation consists of two quantities or variables whose values are linked, such that changes in one correspond to (possibly) changes in the other in a smooth continuous manner (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). For simplicity we use the term "variable" in this paper to encompass both real-world quantities and mathematical variables. Covariational reasoning consists of mental actions related to situations involving covariation, which have generally been described through broad "behaviors" and developmental "levels." For example, Carlson et al. (2002) described covariational reasoning through coordinating variables' values, coordinating directions of change, coordinating amounts of change, and coordinating changes in the rate of change. Thompson and Carlson (2017) gave an alternate set of broad levels, including asynchronous tracking of variables' changes, coordinating variables' values, envisioning changes in chunks, and envisioning changes smoothly (see also Castillo-Garsow, 2012).

Covariational reasoning has been a powerful framework for building coherent mathematics instruction. It has been used to reimagine key ideas such as function (Ellis et al., 2016; Oehrtman et al., 2008; Paoletti & Moore, 2018), graphs (Carlson et al., 2002; Moore & Thompson, 2016; Rodriguez, Bain, Towns, et al., 2019), coordinate systems (Moore et al., 2013), and calculus concepts (Jones, 2017, 2019; Kertil & Gülbağcı Dede, 2022; Thompson, 1994). Covariational reasoning also coheres well with quantities-based thinking and reasoning at the intersection of math and science, or with other disciplines. It has been used to describe reasoning in chemical kinetics (Rodriguez, Bain, Towns, et al., 2019), in scientific representations (Panorkou & Germia, 2021), and in modeling quantities (Panorkou & Germia, 2021).

What might be the next step in building covariation as a construct in educational research? A major theme in the literature cited here is that covariation is generally considered through "large-grained" cognitive structure, such as a student's ability to coordinate directions of changes, or a student's ability to imagine smooth variation. Accounts of "fine-grained" cognitive structure in this area are only beginning to be produced (Rodriguez, Bain, & Towns, 2019; Rodriguez, Bain, Towns, et al., 2019). The next important step is to systematize this emerging "fine-grained" work to better understand the nature of actual covariational reasoning used by students in practice.

Knowledge-in-Pieces and Coordination Class Theory

Overview

Knowledge-in-pieces (KiP) (diSessa, 1988, 2004, 2018; diSessa & Sherin, 1998) is a paradigm centered on fine-grained knowledge and conceptual change. KiP is based on a manifold view of cognitive structure, where knowledge is modeled as an emergent system of elements that are activated in concert, in response to context and perceptual cueing. Knowledge elements can be quite small, such as intuitive extractions from experience (e.g., *closer means more intense*), or a simple idea picked up from a class (e.g., *x means a variable*). These individual knowledge elements become *resources* that a person can activate in a situation (Hammer, 2000). Thus, we call them *knowledge elements* and *resources* interchangeably.

KiP and Graphing

Because much work in covariation has been done in the context of graphing (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Moore et al., 2019; Moore & Thompson, 2015), we provide here some work done on graphing using a KiP lens. Recently, Rodriguez and colleagues (Rodriguez, Bain, & Towns, 2019; Rodriguez & Jones, 2024; Rodriguez et al., 2020) have interpreted students' graphical work through a fine-grained examination of knowledge resources students might flexibly use at different times. This work resulted in a large collection of "graphical forms" students might employ such as *curved means changing rate*, *big dot as focal point, discontinuity means sudden*, and so forth (see Rodriguez & Jones, 2024 for more). As students create a graph, they might infer graphical patterns based on what they observe in the context. For example, if one is creating a graph to model the height of grass mowed once a week, they could notice the sudden change in grass height when cut. This could lead to the activation of *discontinuity means sudden* and they might draw a jump discontinuity in the graph. However, in another situation, there may be no sudden events, so no inference is made to *discontinuity means sudden*. Thus, not observing a resource in context does not imply the students does not posses it.

In addition to graphical forms, there are also more general knowledge elements regarding axes, scaling, beliefs, conventions, and quantities (Jones & Rodriguez, 2023). These can include resources like *spacing means duration* on an axis, the belief that *the presence of π implies trig functions*, or the quantitative resource *time as a natural independent variable*. When students are asked to create graphs, they extract information from the context that traces inference to some of these resources, which are activated and in turn may trace inferences to other resources. This activated collection of resources are used in making moves toward constructing a graph. Graphical resources play an important role as elements nested within larger knowledge systems such as a coordination class, and have strong connections to covariational reasoning, as well.

Coordination Class Theory

Coordination class theory builds on knowledge-in-pieces to provide a model to describe a "concept" within a systems view of knowledge (diSessa & Sherin, 1998). Importantly, not every concept can be modeled as a coordination class, with a coordination class being a specific type that serves the function of allowing an individual to recognize or "see" the concept across contexts. To illustrate, consider the idea of "proportional reasoning." The actual cognition pertaining to proportional reasoning is complex, with basic ideas of difference, products, scaling, balancing, ratios, rates, fractions, inverse relationships, and so on, networked together in some way (Jacobson & Izsák, 2014; Sherin, 2001). Here, a sophisticated coordination class would allow an individual to recognize and apply these ideas across a variety of situations where proportional reasoning is relevant, such as in constant difference, constant growth, or exponential growth contexts. Due to context, it is not necessary to employ the entire coordination class, and only a subset of resources might be activated in a situation. For example, a constant difference situation might elicit a distinct set of knowledge resources related to proportions than an exponential growth situation. The functioning of the concept in one specific situation (subset of resources used) is called a *concept projection* (diSessa, 2004; diSessa & Wagner, 2005).

The defining structural features of a coordination class that allow an individual to see a concept involve two major components: *extractions* (previously called *readouts*) and *inferential nets* (diSessa, 2018; diSessa & Sherin, 1998; diSessa & Wagner, 2005). Extractions are the features attended to – what are seen, noticed, or observed in the context that are relevant to the target information. In the proportion example, features that may cue a student to recognize proportionality in a situation might be specific symbol templates (A/B = C/D), the presence of rates (X lbs/min), percentages (X %), and so on. Inferential nets are the set of possible inferences that lead toward the target idea. These can connect an extraction to a knowledge element, or may connect that knowledge element to yet additional knowledge elements.

Covariation as a Coordination Class

We now turn to the main purpose of this paper, which is to propose covariation as a coordination class, and covariational reasoning as the employment of a covariation concept projection into some context. In the current literature, "covariation" levels tend to be given as hierarchical progressions toward the ideal of covariation. While this lays a useful and important foundation, we posit that we can zoom in further into the concept of "covariation" by modeling it as a coordination class. Here, it would consist of a large network of resources that function differently in different situations, such as when constructing a graph, or identifying quantitative relationships in a physics equation, or imagining different types of mathematical functions. Different collections of resources pertaining to covariation might be activated in certain of these situations, but not in others, meaning each situation represents a *concept projection* of covariation into that context. For each situation, one would employ extractions to "read" the situation and begin making inferences across the knowledge elements, to determine the desired information. For example, in a graph-interpretation context, extractions might consist of observing features in the graph's shape, noticing axis scales or labels, or locating straight or curved sections of the graph. In a context considering a scientific quantitative relationships, extractions might consist of how those real-world quantities behave, or how their magnitudes change. In a math-function context, extractions might consist of noticing what type of function is present, or what symbols portray the input or output. Inferential nets connect these extractions to knowledge elements that might be relevant for reasoning covariationally. These might include resources like *straight means constant rate*, or *up-down as inverse relationship*, or *sine as periodic*. Inferences can further be made from these resources to yet other resources, in a chain that activates a certain subset of knowledge elements in that context.

To dig deeper, let's consider a common example of a covariational situation: the well-known bottle task (Carlson et al., 2002; Johnson, 2015a, 2015b). In this task, a bottle is filled with water and students are asked to draw a graph of volume as a function of height. Analyzing this task has previously consisted of finding evidence for certain levels of covariational reasoning through students' behaviors (Carlson et al., 2002), or describing general natures of the coordination (Johnson, 2015a, 2015b). Certainly these findings have been very helpful in understanding students' thinking. To now reframe this situation using coordination class theory, we first note that students' covariational reasoning for this task reflects a single concept projection into the bottle context. Thus, we should expect that subsets of their resources might be activated, while other resources remain dormant. In this concept projection, the extractions would be those features students see or notice about the bottle that would be relevant (to them) in identifying the covariational relationship and capturing it graphically. These might include noticing a constant rate of water pouring in, the width of the bottle, or the emptiness of the bottle of at the beginning.

An important implication of thinking of covariation as a coordination class is that these extractions then influence what the student might do next, as inferences are made to other knowledge elements. Their extractions drive inferences for which resources get activated. It is entirely possible that seeing the constant rate of pouring might activate a resource that *straight means constant rate*, leading them to initially draw a straight line segment. In coordination class theory, this action does not reflect a misconception (Smith et al., 1994). This resource is perfectly legitimate in certain contexts; it is simply misapplied here, because that volume per time rate is not what the volume-height graph is meant to reflect.

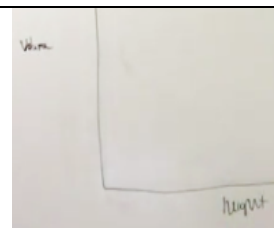
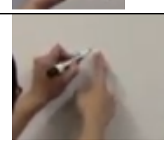
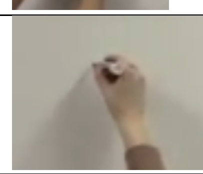
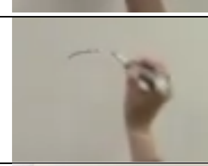
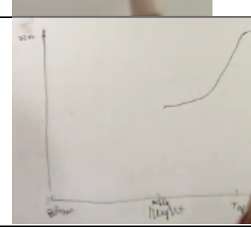
Additionally, what has been termed covariational reasoning "levels" (Carlson et al., 2002; Thompson & Carlson, 2017) may be reinterpreted as possible extraction strategies students use in their concept projection. For example, gross coordination (i.e., general increase/decrease behavior in the variables), can be a strategy for what to focus on, guiding what students may "see" in the context. Zooming in further, these extractions shape what resources get activated, and the collection of activated resources drives the actions taken. In the bottle context, a student might observe that as water comes in, height and volume both go up, leading to the activation in the inferential net of *up-right means increasing*. Consequently, the students may make a general up-right initial sketch. Increasingly sophisticated covariation levels may represent increasingly sophisticated methods for reading situations for relevant information. But if we view covariational reasoning simply as a set of levels, the in-the-moment cognition induced from that extraction strategy and the subsequent inferences might be overlooked.

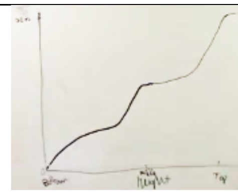
Illustrative Vignette and Discussion

To show covariation-as-a-coordination-class in action, we provide here an illustrative example of students working in the well-known bottle context and use it to suggest the power one can get from the coordination class perspective. Figure 1 shows an excerpt from two students working with a version of the bottle task, and includes their speech and inscriptions. We also document the knowledge resources activated on the right of the figure (see Jones & Rodriguez, 2023; Rodriguez & Jones, 2024 for more details on these specific resources).

Figure 1. Eric/Ellie's work in the bottle task, with identified extractions (E), resources (R), and other inferences (I)

Task: This bottle is being filled up with soda. Graph the volume of the soda in the bottle as a function of the height of the soda.

Name	**Utterance**	**Drawing**	**E=extraction** **R=activated resource** **I=other inference**
Eric:	I'm going to assume it fills up, like the volume.		E: Volume fills up
Ellie:	Let's start at zero. So, which axis should be volume?		E: Bottle starts empty R: Origin as natural start R: Axes need labels
Eric:	Probably this one [points to y-axis], 'cause the height [points to x-axis] is the constant increase. … I think it's a more, I'm guessing this is probably a curved function [i.e. graph], because the bottle's curved… It'd probably fill up the volume if you're going up height.		E: Height goes up R: Independent as 'marching on' R: Other on vertical axis E: Bottle's curvy shape R: Graphs preserve visual features I: graph is curvy
Ellie:	Like that, a little bit, maybe	Gestures 'up-right' arc	R: Up-right as increase
Eric:	… It would take up longer to fill up the areas where the bottle is wider.	Gestures fingers apart 	E: Bottle shape R: More takes longer
Eric:	…[later]… Like in the middle we should probably have a point [touches middle of graph space], kind of like right here [draws a dot in the middle].		E: Halfway point as significant R: Dot as focal point
Ellie:	Yeah. I feel like the closer we get to the top [draws a dot at the top-right],		E: Top of bottle I: scenario ends at top R: Dot as focal point
Ellie:	[continued] the more it'll kind of level off a little bit [draws slight curve going into a straight line segment].		R: Plateau as leveling off
Eric:	'Cause it's more constant [i.e. straight line segment]. This is like the middle of the bottle, so it's like a steady followed by a rapid increase. [Ellie agrees: "yeah"].		R: Straight means constant rate E: Middle is curvy I: Growing rate of increase R: Connection as transition R: Steepness as rate

Eric:	And then I think it's the same thing here [points to left of graph portion]. It's kind of followed by a sharp kind of, just kind of like gradual [finishes drawing curve].		R: Connection as transition R: Steepness as rate

In a "holistic cognition" analysis, we might identify that the students can perform basic coordination (e.g., labelling axes) and gross coordination (e.g., up-right graph), and can coordinate values (e.g., identifying height-volume pairs for the middle and top) and coordinate rates (e.g., less steep to more steep) (Figure 2, left). However, stating these does not help distinguish this pair of students from another pair who might use similar levels, but in a very different way. For example, in constructing their graph, Ellie and Eric did not move in an orderly "left-to-right" fashion, which is not accounted for in major covariational reasoning frameworks. They chose to start at the middle and end, but what accounts for those actions? Similarly, a holistic view might identify a difficulty in constructing the middle of the graph, and might try to identify the level at which the students struggle, such as the "changing rates" level from Carlson et al. (2002). But Ellie and Eric successfully manage changing rates in other places, so this description does not seem to provide much detail on the actual cognition being used.

By contrast, what might a coordination class interpretation entail? As mentioned before, it provides the ability to "zoom in" on each of these covariational reasoning levels to see how those extractions lead to other inferences, including graphical forms, graphical resources, or other knowledge elements (Figure 2, right). In Figure 3, we provide exactly this type of "zoom-in" in the form of a resource graph that models at a much more detailed level what these students' cognitive work consisted of. Notice that the covariation "levels" functioned more as broad strategies for extracting information from the task, leading to resources activation. This perspective allows us to see how their reasoning at each level operated in-the-moment. It could allow us to distinguish Ellie and Eric's work from other students' work using similar levels.

As they extracted information, they connected it to relevant resources and used those to build certain pieces of the graph, one at a time (Figure 3). They started by constructing axes they felt matched the task, based on resources related to the origin, labels, and what counts as an "independent" variable versus the "other" variable. After identifying the general "up-right" nature of the graph, they identified key locations in the graph, such as the mid-point and end-point, based on graphical forms related to dots, plateaus, and straight segments. They progressed to generating curves for the later and earlier portions of the graph, based on graphical forms related to slope, curves, and connecting points (Figure 3). When these two sections did not match up, they simply invoked a belief about the necessary "curviness" of the graph, and a graphical form *curve means changing rates* to try connect them. Thus, the incorrect middle portion seems less about abilities at the large-scale covariation levels, and more about which resources they invoked at that particular time to address a perceived discrepancy.

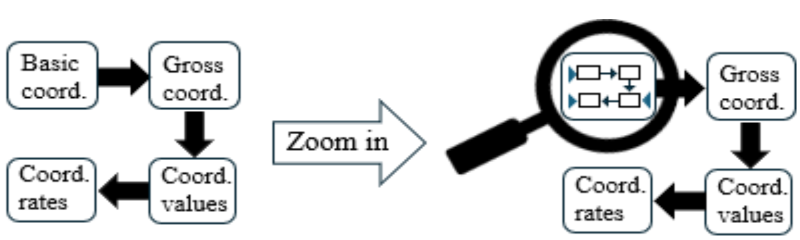


Figure 2. Coordination class theory helps with more explicit "zooming" into covariational levels (see also Figure 3)

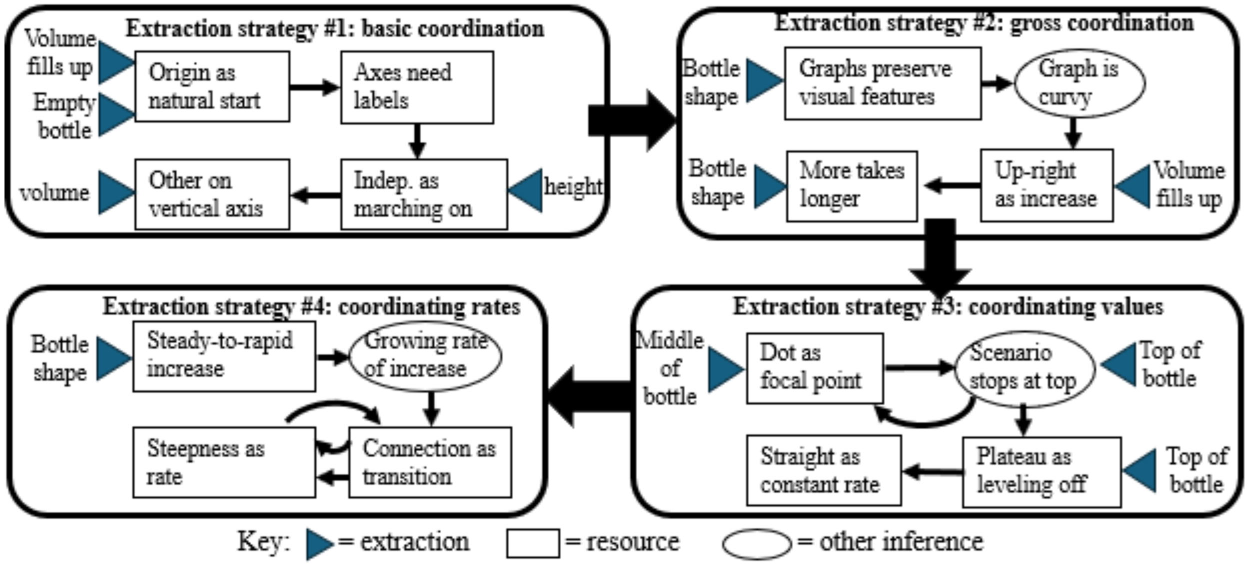


Figure 3. Resource graph as a model of the students' cognition with much greater detail

Further, consider how they determined to draw curves for a certain portions. Sometimes it was driven by an inference from the increasing rate ("it's a steady followed by a rapid increase"). Other times it was driven by the visual appearance of the curvy bottle ("probably a curved function, because the bottle's curved"). Positioning their reasoning at a certain level does not account for this subtle shift in justification. A coordination class lens helps us move away from viewing such ideas as unitary objects, toward seeing more fluid reasons and justifications.

Also, it is possible some might want to deem it "wrong" that the students appeared to use the bottle's curvy appearance at one point to infer that the graph was also curvy. But, again, the coordination class perspective helps us realize that that idea might sometimes be a legitimate inference in certain contexts, but perhaps not for others. In fact, the incorrect part in the middle is likely due to an overuse of this resource, in that they extracted the curviness of the bottle, combined it with the fact that changing rates are reflected by curves, and produced a tentative sketch of a graph they felt depicted that. This lens provides better instructional fodder, as well, because an instructor can try to highlight what resource the students drew on in that particular instance and to help the students question whether it was the right one to use. In other words, the difficulty might not be due to some larger underlying issue with one of the covariational reasoning levels. Rather, it could be as simple as the activation of the wrong resource at the wrong time. Helping the students identify their knowledge usage could help them shift from that resource to other resources, such that they could correct their incorrect portion of the graph. This constitutes, in part, the gradual shift from more novice activity to more expert activity.

Conclusion

In summary, we believe we get significant power through a coordination class interpretation in modeling these students' cognition related to covariational reasoning and to account for individual actions taken. It also provides a better narrative that follows what the students actually do. We also have better recourse for assisting the students in developing expertise, by shifting their resource usage and helping reorganize their knowledge elements. This power is important to researchers, as it improves our ability to faithfully render actual student thinking in-the-moment in real time. This power is also useful to instructors, in our ability to track what our students are thinking and doing in any given moment, in order to better scaffold and direct it.

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