Integration by Substitution: A Justification Based on Multiplication, Measurement, and Inverse Proportions

Andrew Izsák
Tufts University

Researchers have argued that interpreting definite integrals and their notation as summations of small amounts of a target quantity (Adding Up Pieces) is especially productive for applied problems. Jones and Fonbuena (2024) extended research on Adding Up Pieces by providing an analysis of integration by substitution grounded in quantitative reasoning. The present theoretical paper develops a complementary justification that makes three contributions. First, whereas past research on integration has rarely foregrounded explicit, quantitative meanings for multiplication, the analysis presented here is based on connections between multiplication and coordinated measurement with two units. Second, the analysis presented here uses coordinated measurement to explain how, across diverse situations, small amounts of a target quantity can be understood as products. This supports reasoning about Riemann sums and areas as approximations for definite integrals. Finally, the analysis provides a new justification for integration by substitution that relies on rectangular areas and inversely proportional relationships.

Keywords: Calculus, Integration, Multiplication, Measurement

Introduction

A growing body of research has reported on students' interpretation and construction of integrals, especially when solving problems in physics and other applications outside of pure mathematics (e.g., Cui et al., 2006; Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Thompson, 1994a). Recently, Jones and Fonbuena (2024) pointed out that this literature has largely overlooked the central topic of integration by substitution, or u-substitution. They then developed a conceptual analysis of the topic grounded in Thompson's account of quantitative reasoning (e.g., Smith & Thompson, 2007; Thompson, 1990) but, in that analysis, stopped short of discussing explicit, quantitative meanings for multiplication. The present report develops a complementary analysis based on (a) connections between multiplication and coordinated measurement with two different units and (b) inversely proportional relationships. The results add to a growing body of studies that have examined how quantitative interpretations of multiplication based on coordinated measurement can support calculus learning (e.g., Beckmann & Izsák, 2024; Izsák, 2023)

The paper is organized into four sections. In the first, I give a brief review of research on definite integrals that includes a summary of the results reported by Jones and Fonbuena (2024). A main point will be that past research has often referred to multiplication in connection with the $f(x) \bullet \Delta x$ and $f(x)dx$ notations without articulating an explicit, quantitative meaning for the operation. In the second, I give a brief overview of research on multiplication with quantities and point to complexities highlighted in that research that have yet to be grappled with directly in research on definite integrals. In the third, I review recent work on a coordinated measurement meaning for multiplication that can provide coherence across a wide range of problem situations and topics, including situations that often serve as applications of integration. A main point will be that the coordinated measurement meaning for multiplication can support using Riemann sums and rectangular areas to approximate definite integrals across diverse situations. In the fourth, I explain how the coordinated measurement meaning for multiplication can support a justification for integration by substitution that is based on Riemann sums and inversely proportional relationships.

Review of Research on Definite Integrals

Applying definite integrals to solve problems about physical situations requires reasoning in terms of quantities, such as lengths, time, work, electric charge, etc. Past research has provided a variety of perspectives on such reasoning, and much of it has been grounded in Thompson's account of quantitative reasoning (e.g., Thompson, 1994b; Thompson, 2011). One line of work has focused on different interpretations of definite integrals and their notation. Thompson and colleagues (e.g., Thompson et al., 2013; Thompson, 1994a; Thompson & Silverman, 2008) have argued that interpretating integrals as accumulations from rates of change can facilitate reasoning about the Fundamental Theorem of Calculus. In other work, Jones (e.g., 2013, 2015) has found that college students can interpret the $\int_a^b f(x)\,dx$ notation as indicating addition of small pieces (Adding Up Pieces), denotating boundaries of a region in the plane, and signifying a function whose derivative is the integrand. At the same time, researchers have debated whether or not, when using Adding Up Pieces, students should understand those pieces as composed through multiplication (e.g., Ely, 2017; Jones et al., 2017; Jones & Ely, 2023; Thompson & Harel, 2021).

A second set of perspectives has decomposed reasoning about definite integrals into layers that include ones focused on local relationships (e.g., Oehrtman, & Chhetri, 2015; Oehrtman, & Simmons, 2023; Seeley, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). As one example, Sealey articulated the *Riemann Integral Framework*, which is a mathematical decomposition of Riemann sums and definite integrals into five layers that she called the Orientation, Product, Sum, Limit, and Function layers. She reported that university calculus students had the most trouble working in the Product Layer, which is where students interpret the $f(x)dx$ notation. As a second example, Oehrtman and Simmons proposed *Emergent Quantitative Models* as a tool for tracing how students' attention shifts across three scales—named the Basic, Local, and Global models—when constructing definite integrals. A Basic model relates quantities with constant values, a Local model restricts a basic model to small regions where varying quantities are approximately constant, and a Global model is a sum based on Local models. Thus, similar to the Product Layer, the Local model is where students would reason about products of the form $f(x) \bullet \Delta x$.

Research on definite integrals summarized above has consistently highlighted the importance of reasoning about small pieces of a target quantity but, to the best of my knowledge, Jones and Fonbuena (2024) were the first to analyze such reasoning in the context of integration by substitution. They began their conceptual analysis by decomposing Adding-up-Pieces reasoning into three aspects—*partitioning* a quantity into small pieces (precursors to differentials), *determining the target quantity* in each of those small pieces (similar to the Product Layer and the Local model mentioned above), and *summing* the small pieces of the target quantity to get the total quantity (similar to the Global model mentioned above). Their analysis traced the consequences for each of these three aspects when one changes from one input variable to another. They described these consequences as "conversions" of the partition, integrand quantity, and interval of summation.

Although past research on definite integrals has frequently attended to small intervals where relationships between quantities are approximately linear, to the best of my knowledge, none of these studies has addressed explicitly an associated, quantitative meaning for multiplication. The next section explains why this omission is consequential.

Review of Research on Multiplication with Quantities

In comparison with the literature on definite integrals, theoretical and empirical research on multiplication is much older. For the present analysis of integration, I concentrate on theoretical results. Theoretical perspectives on multiplication have included classifications of problem situations in which one typically multiplies or divides (e.g., Greer, 1992; Nunes & Bryant, 1996; Vergnaud, 1983, 1988); interpretations, or meanings, of multiplication as an operation on quantities (e.g., Davydov, 1992; Izsák & Beckmann, 2019; Schwartz, 1988; Thompson & Saldanha, 2003); and psychological primitives that include repeated addition (Fischbein et al., 1985), splitting (e.g., Confrey, 1994), and units coordination (e.g., Hackenberg et al., 2023; Hackenberg & Tillema, 2009; Steffe, 1988; 1994). The analysis of integration by substitution presented below makes the most direct contact with the strands focused on classification and on interpretations of multiplication as an operation on quantities.

For classification research, I highlight Vergnaud's (1983, 1988) seminal contribution. He has analyzed mathematical structures for multiplying with quantities and distinguished two primary subtypes. *Isomorphism-of-measure-spaces* (I-o-M) situations are ones where the multiplicative structure consists of a simple direct proportion between two measure spaces, M_1 and M_2 (1983, p. 129). Here the direct proportion is often viewed as a rate of change. *Product-of-measure-spaces* situations (P-o-M) are the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). In anticipation of approximating integrals, notice that Riemann sums use rectangular areas, associated with P-o-M situations, to model both I-o-M situations (e.g., integrating velocity to get distance) and P-o-M situations (e.g., integrating force to get work). One response would be to abandon using rectangular areas as generic models for products. Indeed, some researchers (e.g., Jones & Ely, 2023; Thompson et al, 2013) have claimed that the area-under-a curve interpretation for definite integrals is insufficient for modeling physical situations. An alternative response, pursued in subsequent sections of this paper, is to seek a meaning for multiplication that spans I-o-M situations and P-o-M situations. Such a meaning could help students understand how rectangular areas and, thus, Riemann sums can be used to express products and approximate definite integrals across both I-o-M and P-o-M situations.

Research on interpretations, or meanings, of multiplication as an operation on quantities has consistently focused on the role of units and, by implication, measurement. Izsák and Beckmann (2019) organized these interpretations into two categories, those that emphasize unit transformations and those that emphasize measurement. In one example of the unit transformation approach, Schwartz (1988) argued that when addition is used to model situations both addends and the sum refer to the same unit (e.g., 2 miles + 3 miles = 5 miles) but, when multiplication is used, the units of the factors are transformed into units of the product (e.g., 2 miles per hour • 3 hours = 6 miles). In a second example of the unit transformation approach, Thompson and Saldanha (2003) stated: "By convention, the standard unit of the product quantity is defined as that amount made by one unit of each constituent quantity" (p. 103).

Measurement might be implied in unit transformation approaches but is addressed more explicitly in further quantitative interpretations of multiplication. Thompson (1994b) articulated the following definition of a quantity when introducing his theory of quantitative reasoning: A quantity consists of "an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). This definition implies that numbers are outcomes of measurement: They express how many units, or how much of a unit, exhaust the extent of the "quality" in question. We often use the term "multiplication" to refer to this fusion of number and measurement—for instance, we might say that a taller building is three times the height of a shorter building when what we mean is that three of short height exhaust the tall height. Such multiplicative language, based on measurement with a single unit, is often taken as evidence of "multiplicative thinking." In such cases, measurement, numbers, and multiplication are intertwined.

Multiplication as Coordinated Measurement and Riemann Sums

Further research on quantitative meanings for multiplication has emphasized coordinated measurement with two different units (e.g., Beckmann & Izsák, 2015; Davydov, 1992; Izsák & Beckmann, 2019). Davydov argued that the essential situation for multiplication is one where it is impractical to measure some quantity of interest with a given unit because the unit is much smaller than the quantity one wishes to measure. The solution is to coordinate the smaller unit with an intermediate unit purposefully chosen to measure the quantity of interest more conveniently. As an example, if one wished to determine the number of teaspoons in a volume of milk, one might use cups as an intermediate unit, determine how many teaspoons are in 1 cup, and determine how many cups are in the target volume. Here the two measurement units for the target volume are teaspoons and cups.

Inspired by Davydov (1992), Izsák and Beckmann (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have explicated a coordinated measurement meaning for multiplication. Figure 1 shows an equation where N, M, and P are each outcomes of measurement, as discussed in the previous section. The product amount is measured simultaneously in base units, P base units, and in group units, M group units. N expresses the measure of 1 group unit in terms of base units. Consider again our milk example, where a volume is measured in both teaspoons (base units) and cups (group units). If 48 is the measure of 1 cup in terms of teaspoons, and 7 is the measure of the target volume in terms of cups, then 336 is the measure of the target volume in terms of teaspoons. Following the meaning in Figure 1, our $N \bullet M = P$ equation would be 48 (teaspoons in 1 cup) • 7 (cups in 1 target volume) = 336 (teaspoons in 1 target volume).

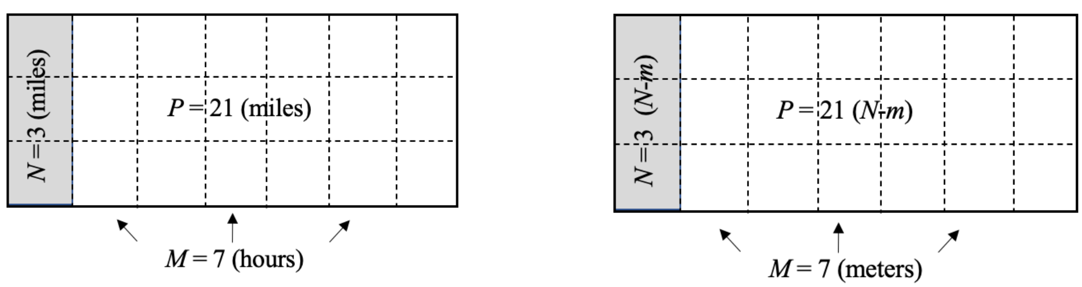
N	•	M	=	P
How many base units make one group unit?		How many group units make one product amount?		How many base units make one product amount?

Figure 1. A coordinated measurement meaning of multiplication (Izsák & Beckmann, 2019, p. 91)

Izsák and Beckmann (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have pointed out that the coordinated measurement meaning for multiplication works with positive real numbers, can tie together diverse situations using a single shared quantitative structure, and challenges the field to rethink several established results. One of those results is the partition of multiplication situations into structurally distinct subclasses—for instance I-o-M and P-o-M situations.

Figure 2 shows how the coordinated measurement meaning for multiplication summarized in Figure 1, combined with rectangular area, can fit both I-o-M and P-o-M situations. The same rectangular area represents a product amount measured both in base units (each small square represents 1 base unit) and in group units (each column represents 1 group unit). One can think of these situations as Basic models (Oehrtman & Simmons, 2023) for two common applications of integration. Figure 2a fits a problem about change in location (a walked distance). If each small square represents 1 mile (1 base unit) and each column corresponds to 1 hour (1 group unit), then the measure of one column is 3, the change measured in hours is 7, and the change measured in miles is 21. The assignment of units to *N*, *M*, and *P* is in contrast to the more usually assignment of units: 3 miles per hour • 7 hours = 21 miles. Figure 2b fits a problem about change in energy. If each small square represents 1 Newton-meter (1 base unit) and each column corresponds to 1 meter (1 group unit), then the measure of one column is 3, the change measured in meters is 7, and the change measured in Newton-meters is 21. Again, this assignment of units to *N*, *M*, and *P* is in contrast to the more usually assignment of units: 3 Newtons • 7 meters = 21 Newton-meters.

The examples in Figure 2 indicate how attention to two measurement units, made explicit in the $N \bullet M = P$ equation, allows one to see parallels (i.e., there is an isomorphism) between rectangular areas and the two applied situations, one an I-o-M situation and one a P-o-M situation. The same reasoning demonstrated here with a Basic model works equally well with Local models (e.g., Oehrtman & Simmons, 2023). Thus, Riemann sums based on rectangular areas can be used to approximate target quantities in both I-o-M and P-o-M situations. Learning to think about multiplication in terms of coordinated measurement could help students understand how rectangular areas can be used as a generic models for products and why multiplication based on rectangular areas gives the same numerical values as the corresponding multiplication in applied situations. The next section relies on Riemann sums based on rectangular areas to develop a justification for integration by substitution.



Shading corresponds to 1 hour.
Walk is measured in hours (7) and in miles (21).

N	•	M	=	P
3 miles in 1 hour	•	7 hours in 1 walk	=	21 miles in 1 walk

(a)

Shading corresponds 1meter.
Change is measured in meters (7) and in *N-m* (21).

N	•	M	=	P
3 *N-m* in 1 meter	•	7 meters in 1 displacement	=	21 N-m in 1 displacement

(b)

Figure 2. Coordinated measurement and rectangular area. (a) An I-o-M situation. (b) A P-o-M situation.

Coordinated Measurement and Integration by Substitution

The analysis of integration by substitution presented here is based on Riemann sums and, similar to the analysis given by Jones and Fonbuena (2024), is based on a change in the independent variable through function composition. Consider the Moving Truck situation presented below. Speed is taken to be piece-wise constant, and the discussion of Figure 2a above justifies using rectangular areas to represent distances. Figure 3a uses rectangular areas to show distance traveled for each segment of the trip. Think of the graph in Figure 3a as representing a simple Riemann sum.

Three friends are driving a moving truck from Detroit to Boston.

- *Steve takes his turn first and drives 4 hours at a steady speed of 60 miles each hour. At the end of his turn, the friends stop and rest for 1 hour.*
- *Chara takes her turn second and drives 4 hours at a steady speed of 72 miles each hour. At the end of her turn, the friends stop and rest for 1 hour.*
- *Zawadi takes her turn last and drives 4 hours at a steady speed of 54 miles each hour.*

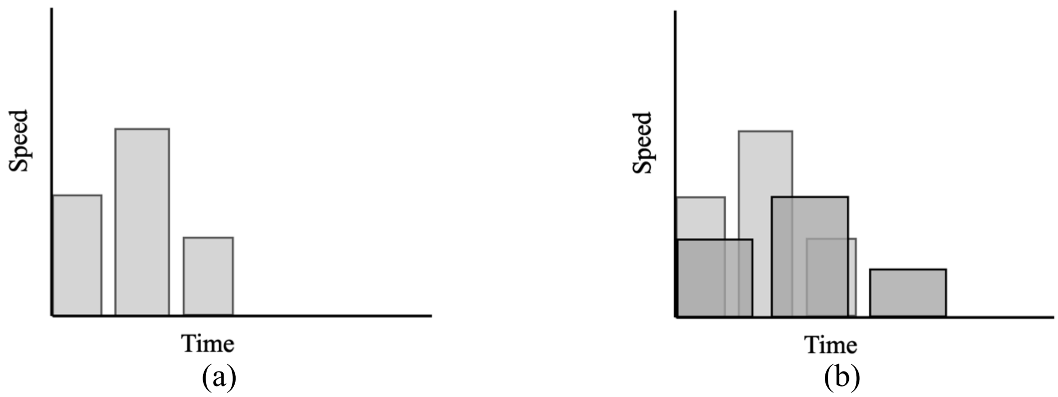


Figure 3.Moving Truck situation. (a) Time measured in hours. (b) Time measured in playlists.

Now suppose that the three friends invent a new unit based on the time it takes to go once through a playlist that they listen to as they drive. Suppose that 1 playlist is 2/3 of an hour. The miles driven in 1 playlist are fewer than the miles driven in 1 hour by a factor of 2/3. Thus, speeds measured in miles per playlist are 2/3 of the corresponding speeds measured in miles per hour. Simultaneously, the number of playlists it takes to measure any given duration of time is 3/2 the number of hours it takes to measure the same duration of time. Figure 3b superimposes onto Figure 3a a second graph of speeds and times for each segment of the trip based on playlists. Think of the graph in Figure 3b as representing a new Riemann sum that expresses distances when time is measured in playlists. Comparing areas when measuring time in hours and when measuring time in playlists, notice that we have an inversely proportional relationship: Shrinking in the vertical direction is offset by stretching in the horizontal direction, the distance for each trip segment measured in miles is the same whether time is measured in hours or in playlists, and the total shaded areas corresponding to the two Riemann sums are the same. Again, reasoning based on a Basic model applies equally well to reasoning based on a Local model.

In the Moving Truck situation, there is a single linear relationship between the original input variable, hours, and the new input variable, playlists. If students used a different unit of time for each segment of their journey, the same argument would work, each rectangle associated with time measured in hours (Figure 3a) would have the same area as the corresponding rectangle with time measured in the new unit. Thus, the Riemann sum depicted in Figure 3a would still have the same area as the transformed Riemann sum depicted in Figure 3b. The statement of integration by substitution is then plausible because the two definite integrals are approximated by the same sequence of values.

Discussion and Conclusion

To the best of my knowledge, the justification for integration by substitution presented here is both original and at odds with more traditional justifications. In traditional calculus courses, integration by substitution is often discussed as the "reversed chain rule," which can be expressed with the following equation: $\int_a^b f'\big(g(x)\big)\, g'(x)dx \;=\; \int_{g(a)}^{g(b)} f'(u)du$. (A more general statement holds where f is continuous.) The student is tasked with recognizing functions f, g, and g' in the integrand on the left-hand side of equation that afford the transformation to the right-hand side of the equation, an integral one hopefully knows how to compute. One common justification for this transformation replaces $g(x)$ with u and $g'(x)dx$ with du and, thus, associates $g'(x)$ with dx.

In contrast to the traditional justification, which is motivated by how integration by substitution is typically applied, the justification presented here started with a Riemann sum for $\int_{g(a)}^{g(b)} f'(u)du$ and explained why the corresponding Riemann sum for $\int_a^b f'\big(g(x)\big)\, g'(x)dx$ gave the same number. This new justification requires reexamining several common ways of reasoning about integration. I have already discussed how the coordinated measurement meaning of multiplication supports using rectangular areas as generic models for products that span both I-o-M and P-o-M situations. In addition, the new justification consistently interprets integrands as rates. In particular, rather than associating $g'(x)$ with dx, as is common in more traditional explanations, the justification presented here associates $g'(x)$ with $f'(g(x))$ and the entire $f'(g(x))\, g'(x)$ expression is treated as a rate of change. For the Moving Truck example, $g'(x)$ rescaled rates of change represented by the vertical dimension that were caused by a change in the units used for measuring time. This, in turn, supports consistent interpretations of integrands across related topics, including the Fundamental Theorem of Calculus. Finally, although the new analysis is not a proof, it highlights how inverse proportions, a topic from middle grades mathematics, makes the statement of integration by substitution plausible. It is not a rigorous explanation *that* the statement is true but, rather, offers a plausible explanation for *why* it is true.

The analysis of integration by substitution provided here extends recent research on ways that multiplication understood as coordinated measurement with two different units can support coherent pathways from elementary and middle grades topics to calculus topics (e.g., Beckmann & Izsák, 2024; Izsák, 2023). As one example, Beckmann and Izsák argued that the large majority of first-year calculus is the study of situations in which *limiting behavior of co-variation takes place within a structure that coordinates multiplication with two measurement units*. They highlighted two different perspectives on proportional relationships, multiple batches and variable parts, and argued that approaches to rates of change based on multiple batches and unit rates are ill-suited for understanding $\frac{\Delta x}{\Delta y}$ as a single measure that remains (approximately) constant as the differences Δx and Δy tend to zero. Beckmann and Izsák then explained the concomitant affordances of variable parts and measurement rates. Finally, by attending to quantitative meanings for multiplication as a foundation for calculus, the present report resonates with other recent research that has examined sources of students' difficulties with calculus that reach back into middle grades topics (e.g., Boyce et al., 2021; Byerley, 2019; Carlson et al., 2015; Thompson & Harel, 2021). This emerging body of research suggests that a substantial reexamination and revision of how topics related to multiplication are developed in earlier grades could help students be better prepared for studying calculus grounded in quantities.

References

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17–38. https://doi.org/10.5951/jresematheduc.46.1.0017

Beckmann, S., & Izsák, A. (2024). *A variable-parts perspective on foundations for calculus: Rates and slopes as measures*. Manuscript submitted.
Boyce, S., Grabhorn, J., & Byerley, C. (2021). Relating students' units coordination and calculus readiness. *Mathematical Thinking and Learning*, *23*(3), 187–208. https://doi.org/10.1080/10986065.2020.1771651
Byerley, C. (2019). Calculus students' fraction and measure schemes and implications for teaching rate of change functions conceptually. *Journal of Mathematical Behavior*, *55*. https://doi.org/10.1016/j.jmathb.2019.03.001.
Carlson, M., Madison, B., & West, R. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 209–233. https://doi.org/10.1007/s40753-015-0013-y
Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). State University of New York Press.
Cui, L., Rebello, N. S., Fletcher, P. R., & Bennett, A. G. (2006). Transfer of learning from college calculus to physics courses. *Proceedings of the NARST 2006 Annual Meeting, San Francisco, CA.*
Davydov, V. V. (1992). The psychological analysis of multiplication procedures. *Focus on Learning Problems in Mathematics, 14*(1), 3–67.
Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior*, *48*, 152–167. http://dx.doi.org/10.1016/j.jmathb.2017.10.002
Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.
Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* (pp. 276-295). MacMillan.
Hackenberg, A. J., Aydeniz-Temizer, F., & Borowksi, R. (2023). Middle school students at three stages of units coordination learn to make same speeds. *Journal of Mathematical Behavior*, *72*. https://doi.org/https://doi.org/10.1016/j.jmathb.2023.101085
Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior*, *28*(1), 1–18. https://doi.org/10.1016/j.jmathb.2009.04.004
Izsák, A. (2023). Foregrounding inversely proportional relationships in integration by substitution. In S. Cook, B. Katz, & D. Moore-Russo (Eds.). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (p. 1323). Omaha, NE.
Izsák, A., & Beckmann, S. (2019). Developing a coherent approach to multiplication and measurement. *Educational Studies in Mathematics*, *101*(1), 83–103. doi: 10.1007/s10649-019-09885-8
Jones, S.R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior*, *32*(2), 122–141. http://dx.doi.org/10.1016/j.jmathb.2012.12.004
Jones, S.R. (2015). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior*, *38*, 9–28. https://doi.org/10.1016/j.jmathb.2015.01.001
Jones, S., Lim, Y., & Chandler, K. (2017). Teaching integration: How certain instructional moves may undermine the potential conceptual value of the Riemann sum and the Riemann

integral. *International Journal of Science and Mathematics Education*, *15*, 1075–1095. https://doi.org/10.1007/s10763-016-9731-0

Jones, S. R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education*, *9*(1), 8–35. https://doi.org/10.1007/s40753-022-00203-x

Jones, S. R., & Fonbuena, L., C. (2024). U-substitution through quantitative reasoning: A conceptual analysis. In S. Cook, B. Katz, & D. Moore-Russo (Eds.). *Proceedings of the 26th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 913–921). SIGMAA on RUME.

Nguyen, D., & Robello, N. (2011). Students' understandings and application of the area under the curve concept in physics problems. *Physical Review Special Topics Physics Education Research*, *7*, 010112. https://doi.org/10.1103/PhysRevSTPER.7.010112

Nunes, T., & Bryent, P. (1996). *Children doing mathematics*. Blackwell Publishers.

Oehrtman, M., & Chhetri, K. (2015). The equation has particles! How calculus students construct definite integral models. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds). *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 418–424). SIGMAA on RUME.

Oehrtman, M., & Simmons, C. (2023). Emergent models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education*, *9*(1), 36–61. https://doi.org/10.1007/s40753-022-00209-5

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). National Council of Teachers of Mathematics; Erlbaum

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, *33*, 230–245. http://dx.doi.org/10.1016/j.jmathb.2013.12.002

Smith, J. P., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). Erlbaum.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119-140. National Council of Teachers of Mathematics; Erlbaum.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). State University of New York Press.

Stevens, B., & Jones, S. (2023). Learning integrals based on adding up pieces across a unit on integration. *International Journal of Undergraduate Mathematics Education* (9), 118–148. https://doi.org/https://doi.org/10.1007/s40753-022-00204-w

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. San Diego State University.

Thompson, P. (1994a). Images of rate and operational understandings of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2/3), 229–274.

Thompson, P. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). State University of New York Press.

Thompson, P. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* (Vol. 1, pp. 33–57). University of Wyoming.
Thompson, P., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools*, *30*(1–2), 124–1147. https://doi.org/10.1080/07380569.2013.768941
Thompson, P., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM Mathematics Education*, *53*, 507-519. https://doi.org/10.1007/s11858-021-01270-1
Thompson, P., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Shifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 95-113). National Council of Teachers of Mathematics
Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43–52). Mathematical Association of America.
Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). Academic Press.
Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). National Council of Teachers of Mathematics; Erlbaum.
Von Korff, J., & Rebello, N. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics Physics Education Research*, *8*, 010125. https://doi.org/10.1103/PhysRevSTPER.8.010125