Partitioning Physical Attributes for Riemann Sum Approximations: Underexamined Competency

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Researchers have investigated several interpretations of definite integrals and their notation, and they have recognized that one particular interpretation, Adding Up Pieces (e.g., Jones, 2013), is especially important for applying definite integrals to physics and other domains. This same research has paid less attention to how students partition physical attributes, and thus construct pieces, in ways consistent with constructing Riemann sum approximations and definite integrals. We analyzed how 8 students taking calculus-based physics courses constructed Riemann sum approximations and integrals for one task about water pressure on a tank wall. All of the students recognized that integration was relevant, but half did not partition the wall in ways compatible with constructing Riemann sum approximations or definite integrals that solved the task. We conjecture that seeing integrands as rates of change and structuring situations in terms of level curves for rates of change is a significant, but underexamined competency.

Keywords: Calculus, Integration, Coordination Class Theory

Introduction

A growing body of research has reported on students' interpretation and construction of integrals, especially when using integrals to solve problems in physics and other applications outside of pure mathematics (e.g., Cui et al., 2006; Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Thompson, 1994a). One key issue is whether and, if so, how students connect definite integrals with adding up small amounts of a target quantity (Ely, 2017; Jones, 2013, 2015). We present interview data in which college students constructed small amounts when approximating a target quantity in one physical situation. We found that even when students interpreted integrals in terms of Adding Up Pieces (e.g., Jones, 2013, 2015), ways that they partitioned physical attributes were not always compatible with forming normatively correct Riemann sums. This implies significant competencies for applying definite integrals to problems outside of pure mathematics that to date have been left underexamined.

In the sections that follow, we review key strands of research on students' reasoning when applying definite integrals to solve problems, primarily from physics. We then summarize our methods, which were based on interviews with eight students who had credit for at least first-semester calculus and who were enrolled in calculus-based physics courses. We then present four cases in which students partitioned a physical attribute, but not in ways compatible with constructing Riemann sum approximations for definite integrals that solved the task. We use these cases to argue that current research has not paid enough attention to competencies for partitioning in anticipation of forming Riemann sums for approximating definite integrals.

Literature Review and Research Questions

Applying definite integrals to solve problems about physical situations requires reasoning in terms of quantities, such as lengths, time, work, electric charge, etc. Past research has provided a variety of perspectives on such reasoning, and much of it has been grounded in Thompson's account of quantitative reasoning (e.g., Thompson, 1994b; 2011). One line of work has investigated different interpretations of definite integrals and their notation. Thompson and colleagues (e.g., Thompson, 1994a; Thompson & Silverman, 2008) have argued for interpretating integrals as accumulations from rates. Jones (e.g., 2013, 2015) has found that college students can interpret the $\int_a^b f(x)\,dx$ notation as indicating addition of small pieces (Adding Up Pieces), denotating boundaries of a region in the plane, and signifying a function whose derivative is the integrand. The Adding up Pieces interpretation is particularly relevant to the present study and consists of (a) partitioning a physical situation into small pieces, (b) approximating the target quantity within each partition piece, and (c) summing the small amounts of the target quantity across all of the partition pieces to arrive at an approximation of the total amount. Further work has considered whether, when using the Adding Up Pieces interpretation, students should understand those pieces as products composed through multiplication (e.g., Ely, 2017; Jones et al., 2017; Thompson & Harel, 2021). Exactly how students partition physical attributes in service of constructing Riemann sums and definite integrals, however, has not been highlighted in these studies.

A second set of perspectives has decomposed reasoning about definite integrals into layers that include ones focused on local relationships (e.g., Oehrtman & Chhetri, 2015; Oehrtman & Simmons, 2023; Sealey, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). As one example, Oehrtman and Simmons (2023) proposed *Emergent Quantitative Models* as a tool for tracing how students' attention shifts across three scales when constructing definite integrals. They named these scales the Basic, Local, and Global models. Anticipating the main example from our results section, a Basic model relates quantities with constant values (e.g., pressure • area = force, $P \bullet A = F$). A Local model restricts a basic model to small regions where varying quantities are approximately constant (e.g., pressure • "small amount" of area = "small amount" of force, $P \bullet \Delta A = \Delta F$). A Global model is an accumulation based on local models (e.g., sum of the small amounts of force, $\sum P \bullet \Delta A = F$). Oehrtman and Simmons made four points. First, students can derive different local models from the same basic model. To illustrate, there are at least two possible local models from the basic $P \bullet A = F$ model, $P \bullet \Delta A = \Delta F$ and $\Delta P \bullet A = \Delta F$. Second, students must make decisions about whether or not to partition. This involves recognizing that variation in values prevents constructing a close approximation using a single, global application of the Basic model. Third, students must decide *how* to partition. Fourth, students must decide whether and how to sum small pieces.

The present study makes closest contact with the third point above. In contrast to Oehrtman and Simmons (2023)—who presented a single example in which students partitioned the plane into concentric circles smoothly and appropriately for the task on which they were working—we found that for eight college students enrolled in calculus-based physics courses partitioning in service of constructing appropriate Riemann sums was not always straightforward. For the present report, we focused on one task and asked: *How did the students partition physical attributes and to what extent was that partitioning compatible with setting up a Riemann Sum and concomitant definite integral?* Although all the students recognized that definite integrals were relevant, we present examples in which they partitioned physical attributes in ways that did not lead to Riemann sum approximations and definite integrals that would solve the task.

Theoretical Framework

Our theoretical frame combines expert interpretation of definite integrals with cognitive components evidenced by students. Vergnaud (1983, 1988) has analyzed mathematical structures for multiplying with quantities and distinguished two subtypes of situations. *Isomorphism-of-measure-spaces* (I-o-M) situations are ones where the multiplicative structure consists of a simple direct proportion between two measure spaces M_1 and M_2 (1983, p. 129). Here the direct proportion can be viewed as a rate of change. One common example is speed • time = distance, where the units might be miles/hour • hours = miles. *Product-of-measure-spaces* situations (P-o-M) are the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). One common example is Newtons • distance = force, where the units might be Newtons • meters = Joules. Riemann sums use rectangular areas to model both I-o-M situations (e.g., integrating velocity to get distance) and P-o-M situations (e.g., integrating force to get work). For the present study, we focus on pressure as a rate (e.g., Newtons per square meter). More generally, if in the $f(x)dx$ notation one interprets $f(x)$ as a rate, then partitioning in anticipation of a Riemann sum involves locating subsets along which that rate is approximately constant. We that researchers have taken different positions on whether one can always interpret $f(x)$ as a rate when modeling situations with definite integral, $\int_a^b f(x)\,dx$.

Our cognitive perspective is informed by Coordination Class theory (e.g., diSessa, 2002; diSessa & Sherin, 1998; diSessa & Wagner, 2005), a strand of theory within the knowledge-in-pieces epistemological perspective (e.g., diSessa, 1993, 2006). diSessa (2002) motivated the notion of coordination classes with the following description of expert accomplishment: "The fundamental assumption behind the idea of coordination classes is that information is not transparently available in the world. Instead, we have to learn how to access different kinds. Indeed, in different circumstances, we may need to use very different means to determine the same kind of information" (p. 43). This description implies that a collection of knowledge resources allows people to see the same concept-specific information across a range of situations, a phenomenon referred to as *span*, and that different subsets of concept-related resources might be used in different situations, a phenomenon referred to as *concept projections* (diSessa & Wagner, 2005; Wagner, 2010). Foreshadowing our concluding discussion, we conjecture that students may need various subsets of resources (an issue of concept projections) to "see" rates and their "level curves" across an expanding range of situations (an issue of span). Moreover, it seems plausible that information about rates of change and level curves are not "transparently available" across both I-o-M situations and P-o-M situations.

Methods

We interviewed five students enrolled in first semester, calculus-based physics (Spring, 2023) and three students enrolled in second semester, calculus-based physics (Fall, 2023). All eight students attended the same selective university in the northeast United Sates. Each student participated in a series of three, one-on-one, semi-structured interviews (e.g., Bernard, 1994; Ginsburg, 1997). The interviews were conducted by the second author and were spaced a few weeks apart. Each interview lasted approximately 1 hour. The students worked on tasks that afforded opportunities to construct approximations through addition of pieces, consistent with constructing Riemann sums. In addition to text, several of the tasks included diagrams or graphs.

We recorded the interviews using two cameras, one to capture the student and the interviewer and one to capture close ups of written work. At the end of each interview, we collected all written work for later analysis. We used a computer to synchronize and combine the two video recordings into a single file, and we transcribed the interviews verbatim.

We analyzed videos for all eight students, attending to their moment-to-moment utterances, inscriptions, and hand gestures. In so doing, it became clear that one task in particular, the Water

Pressure task provided particularly good access to students' construction of approximations in a situation that, for them, was novel. This was in contrast to tasks where students accessed more rehearsed approaches, such as when using sampled velocity to approximate distance traveled. Furthermore, we observed a variety of approaches to partitioning in the Water Pressure task that were not well aligned with constructing normatively correct Riemann sum approximations.

Figure 1 shows the Water Pressure task used in Spring 2023, a task we adapted from Sealey (2014). In contrast to Sealey, we deliberately omitted formulae that might scaffold students' reasoning about multiplication and possible ways to interpret the $f(x)dx$ notation. Because this first group of students appeared to rely on rote recall when relating length and area units, in Fall 2023 we used a width of 4 meters and a depth of 3 feet to see how readily students would measure the tank wall area in meter-feet. One normatively correct approach to the Water Pressure task is to view pressure as a rate: so many units of force per unit of area. A definite integral for computing the total force can then be set up by attending to level curves where the pressure is constant. In this case, those curves are horizontal lines that extend from left to right across the tank wall. As a consequence, the small regions on which to approximate pressure with constant values are thin rectangles that also extend from left to right across the tank wall. Such rectangles were only one of four different approaches we observed when students connected areas and definite integrals. We discuss the three alternative approaches in the next section.

Pressure, P, applied across a surface area, A, creates a total force, F. Consider a vertical side wall of a tank with a width of 4 feet and a maximum depth of 3 feet (see picture below). Assume that the tank is full of water. The pressure on the tank wall increases with the depth of the water according to the following law: $P = 15x$, where x is the depth of the water (the deeper the water, the greater the pressure). Given this information, describe a method for approximating the total force from the water pressure on the wall.

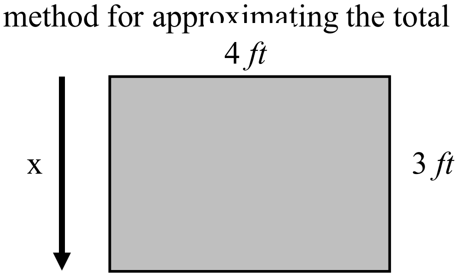


Figure 1. The Water Pressure task

Results

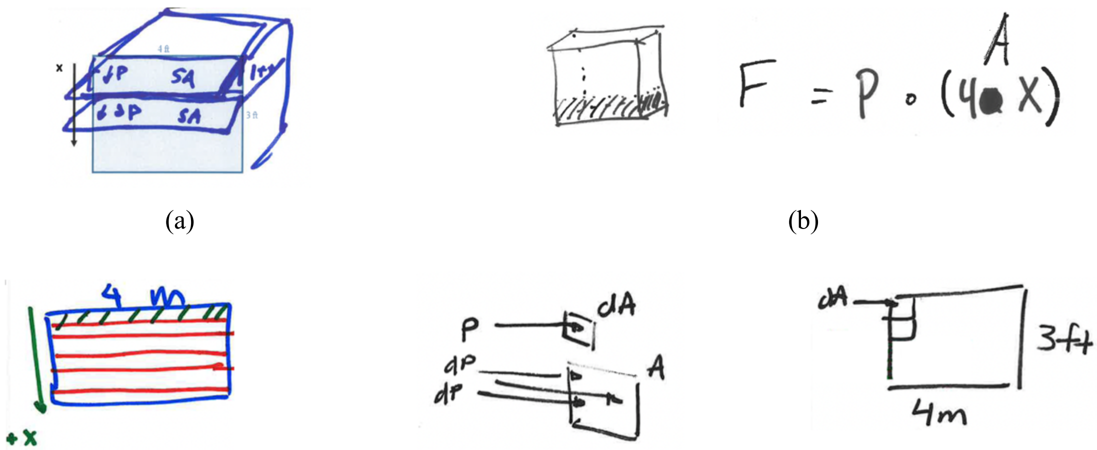
None of the eight students was confident about the relationship between pressure and force, but they all recognized that integration was likely relevant and discussed the $15x$ expression at some point. Thus, they all evidenced knowledge resources spontaneously that we recognized as useful for making progress on the task.

Approach 1: Areas Are Horizonal Cross Sections Parallel to the Bottom

Iliana (SP 23) discussed adding up pieces and located areas as horizontal cross sections of the tank. She read the Water Pressure task and wrote: "To calculate the total force you would take the integral of $P = 15x$ to find the sum of all the surface areas in the tank and evaluate on the bounds (0, x)." She continued by writing "$\int 15x$" and explaining:

Iliana: I just said that to calculate the total force you would have to take the integral of the pressure, and then that is like, integral is really just like finding the sum of all the surface areas in the tank. The like, because like the pressure is just applied across each little area for like x [gestured with open palms to indicate horizontal slices].

Iliana commented "I don't feel like the 4 feet of the width has anything to do with that, but it might. I am not completely sure." When the interviewer asked where Iliana saw surface area, she drew horizontal slices (Figure 2a) and confirmed that her arrows pointed "downward." Thus, her verbal explanation, hand gestures, and drawing gave consistent evidence that she imagined pressure applied to cross sections parallel to the bottom of the tank. Because Iliana may have misread the description of the situation, the interviewer asked whether she considered pressure on the tank wall. Iliana reported that she had not, quickly recognized that "the pressure is like pushing outward onto the side with the force," and stated "But my picture still, this still goes like that [retracing horizontal slices in Figure 2a]." Thus, Iliana's orientation to areas did not shift hand-in-hand with her orientation to pressure.



(a) (b)

(c) (d) (e)

Figure 2. Pieces of area. (a) Iliana, (b) Alex, (c) Melinda, (d) Matthew, (e) Matthew

Approach 2: Areas Are Vertical Regions of the Tank Wall (4*x*)

Alex (SP 23) and Melinda (FL 23) made comments that referenced adding up pieces, and they discussed pressure and area as functions of x. As soon as he finished reading the Water Pressure task, Alex used rate language to describe pressure: "We can say that this up here is pressure, which is in some units of force per square unit area." He considered downward pressure momentarily, similar to Iliana, but then turned his attention to pressure on the vertical wall, as we intended. He explained:

Alex: If we're talking about the pressure out from the water, then what we have to talk about is P times the area of the tank wall. So P, which is $15x$, and then the area of my tank wall is going to be 4 times x [wrote $(4 \bullet x)$, Figure 2b]….So, I want to figure out how much pressure is being applied to that little rectangle [shaded region in Figure 2b]. So, then this would be the pressure applied across the surface area. So, this is my A here [wrote A above $(4 \bullet x)$] creating my total force F on the sidewall of the tank [completed writing $F = P \bullet (4 \bullet x)$].

An exchange later, Alex explained that "my intuition there is if I'm doing something across an area, I usually use multiplication." He then used multiplication to describe the normatively correct relationship between pressure, area, and force:

Alex: It's not that I have a total number P, and I'm splitting it up into smaller chunks. It's that I need to make sure that P is being applied equally to each one of these parts of A. So I would need to be multiplying P by A in order to get that force over the area, or that pressure over the area to find the total force.

Alex's comments left unclear precisely where he located "each one of these parts of A." Although his discussion of parts was disjoint from using a Basic model, he continued using $(4 \bullet x)$ for area as he worked on the task.

Melinda initially wrote $A = 4 \bullet (3 - x)$ and $P = 15x$. She then graphed $P = 15x$, wrote $\int_0^3 15x\, dx$, and explained approximation in terms of adding pressures at different depths: "If we, say, measure at like more points, right? At all these, all these like x values and then we add them all up, then we can get a better approximation of the total pressure that's going against the wall." A few exchanges later, she shaded a rectangular strip at the top of the wall (Figure 2c) and changed her area expression to $4 \bullet (x)$. At the same time, when the interviewer asked Melinda about her $\int_0^3 15x\, dx$ integral, she explained:

Melinda: You would want to find like the area of like a little piece, or like one of these, right. And that would be *dx*, which is the change in *x*, times the height, which is 15 times *x* or like it follows this function. And so, to find the area of that, it's like a rectangle. So, you just multiply the base times the height and then you find the area of that.

Melinda recognized that rectangles on the wall and on her graph of $P = 15x$ referred to different combinations of quantities, but she did not connect local rectangular regions on her graph to local rectangular regions on the tank wall.

Approach 3: Areas Are Regions That Tile But Do Not Span the Tank Wall

Matthew (FL 23) considered two ways to construct a small piece. His first was to consider pressure applied to little bits of area. He wrote $F = \int P\, dA$, a first step toward a normatively correct solution. His second was to consider little bits of pressure applied over the entire tank wall. He then provided a reason for choosing the latter over the former:

Matthew: I don't know if it's going to be area, *dP*, like a little bit of pressure over the entire area, or it's this total pressure over the entire space of the area. But given that the area is constant, and I guess the pressure's changing, I think I would actually not go with this and do, maybe *dP* times *A*.

Next, Matthew wrote $F = \int dP\, A$, plugged in information provided in the task statement, and arrived at $F = 4m \bullet 3\, ft \int_0^3 15\, dx$. A few exchanges later, he reiterated:

Matthew: The reason I said *F* equals *dP* times *A* instead of pressure *dA*, which is the area, is because the, the area isn't changing, like the size of the, the wall isn't changing, but the pressure is with the depth of the water.

Matthew then produced drawings to indicate how he visualized his two approaches (Figure 2d). For the drawing with several arrows labeled "*dP,*" he explained, "And when you, when you sum up all these *dP*s, you would get the pressure." Figure 2e shows how Matthew responded when asked where on the wall he saw *dA*. Notice that he apparently intended to tile the wall, but not with rectangles that spanned the width of the tank wall. A few exchanges later he explained that the pressure was constant on horizontal lines that spanned the width of the tank wall, but he did not revise his approach for setting up an integral.

Approach 4: Areas Are Normatively Correct Thin Slices That Span the Tank Wall (4d*x*)

Three of the remaining four students located a small piece of area as a thin rectangle that spanned the tank wall, even if they encountered other challenges constructing and interpreting integrals to solve the task. Jack (SP 23) quickly drew a horizontal line across the tank wall to indicate a generic small area and wrote the expressions 4 dx. He then used the phrase "pressure over area" and tried to reason about P/A as the integrand, a misdirection that he eventually recognized and corrected. Izsák and Sus (2024) reported on Larisa (SP 23), who generated 15 * .04 = .6N, a normatively correct approximation for a small piece of force at $x = 1$. She then dismissed such a small contribution as inconsequential, in essence questioning the logic of Adding Up Pieces. Torin (FL 23) generated $F = \int_0^3 15x \bullet 4\, dx$, drew lines across the tank wall, but stated repeatedly that dx was simply a notation that did not carry meaning in the situation. In particular, he did not connect dx with a small change in height. Finally, Kathrine (SP 23) generated $\int 15x\, dx$, interpreted this integral as the area under the graph of $15x$, and discussed trapezoids for determining that area, but she made no reference to rectangular areas on the tank.

Conclusion

We examined how students partitioned a physical attribute in service of constructing an approximation in a situation that, for them, was novel. Our results demonstrate that invoking the Adding Up Pieces interpretation of definite integrals (e.g., Jones, 2013, 2015), though often necessary, is not always sufficient for constructing appropriate Riemann Sum approximations for definite integrals. All eight students we interviewed considered using a definite integral for solving the Water Pressure task, and all eight discussed Adding Up Pieces art some point. At the same time, we observed four different approaches to partitioning the tank wall. One approach was normatively correct, but the remaining three were incompatible with constructing approximations for integrals that would solve the task. This result leads us to ask what knowledge would support students' approaches to partitioning physical attributes and generating Riemann Sum approximations, especially for situations with which they are less familiar.

In the Background section, we organized past research on definite integrals into two large strands, one that has identified different interpretations of integrals and one that has delineated multiple layers of attention when generating integrals. Compared to past research, our results suggest that more precise theory is necessary if we are to account for students' reasoning about definite integrals in ways that can ultimately inform instruction. We conjecture that tools from Coordination Class theory could help. Wagner (2016) took a first step in this direction when arguing that students' various interpretations of definite integrals could be understood as different concept projections. Future research could investigate in greater detail how students connect the $f(x)dx$ notation to problem situations. As an example, Matthew's decisions were driven by the fact that the tank wall area was known and constant, which led him away from normatively correct reasoning. Future work should examine how students might look for rates of change and their level curves across diverse situations. As noted by diSessa (2002), relevant information is not always transparently available, and students may need different means to determine how to partition physical attributes when applying definite integrals to solve problems outside of pure mathematics.

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