All My Work is Right, Why Didn't I Get the Right Answer? When Logarithmic Identities Fail

Calum Heldt
University of Nebraska – Omaha

Keith Gallagher
University of Nebraska – Omaha

When learning about logarithms and solving logarithmic equations, students often misapply logarithmic identities. In some cases, misapplying an identity leads to the deletion of solutions or the introduction of extraneous solutions. This results from applying an identity without first checking that all logarithms in that identity are defined on the same domain. In this preliminary work, we examined 14 websites that College Algebra students might use when learning how to solve equations using logarithms, looking for mentions of the domains of definition whenever identities were stated or applied. Of the 11 websites that stated a logarithmic identity, only 4 mentioned domains of definition. Of the 11 websites that showed an example logarithmic identity application, zero mentioned domains of definition. We hypothesize that this lack of emphasis on domains of definition in teaching and learning logarithmic identities may lead to confusion when working with logarithms and in mathematics as a whole.

Keywords: Logarithms, Identities, Domain, College Algebra, Solving Equations

Understanding logarithms and using logarithms appropriately are common stumbling blocks for students, starting in introductory algebra and often continuing into calculus and beyond. Although some research has investigated undergraduate students' understandings of the concept of logarithm (Berezovski & Zazkis, 2006; Leopold & Edgar, 2008; Liang & Wood, 2005; Park & Choi, 2013; Rafi & Retnawati, 2018), little research has focused on undergraduates' knowledge related to logarithmic identities (sometimes also called *logarithm properties* or *logarithm rules*).

Consider the equation $\log(x^2) = 2$. Simply applying the definition of the base-ten logarithm, we obtain that $x^2 = 100$, and thus find that $x = \pm 10$ gives all solutions to the original equation. Checking both solutions in the original equation confirms that both values are indeed solutions to this equation.

However, a common solution path for students taking a course in College Algebra might be to apply the logarithmic identity $\log(m^n) = n\log(m)$ to the original equation to obtain $2\log(x) = 2$. From this point, dividing both sides of the equation by 2 and applying the definition of the base-ten logarithm suggests that $x = 10$ is the only solution to this equation, though we know from the previous example that this equation has at least (indeed, exactly) two solutions.

A student might wonder: "If everything I did was correct, then why didn't I get the right answer?" But the expert observer will note that there is a caveat to the logarithmic identity applied by our hypothetical student: it is true that $\log(m^n) = n\log(m)$, but only when $m > 0$. In particular, it is true that $\log(10^2) = 2\log(10)$, but it is not true that $\log[(-10)^2] = 2\log(-10)$, as the function $f(x) = \log(x^2)$ is defined for all $x \neq 0$, but the function $g(x) = 2\log(x)$ is defined only for $x > 0$. While experts are likely to notice this error, students who are still learning about logarithms and their properties are likely to overlook the solution $x = -10$, and they may even reject it outright due to the fact that one "can't take the log of a negative number."

Our research begins to answer the following research question: How do resources targeted toward College Algebra (or high school algebra) students attend to issues related to function domains when teaching about logarithmic identities? In this study, we explored websites College Algebra students might use as reference materials when learning how to solve problems using logarithmic identities. We focused our attention on the extent to which these websites acknowledge the importance of the domains of the logarithmic functions involved in making a logarithmic substitution during problem solving as in the example presented above. We present some preliminary findings and discuss implications for the teaching and learning of logarithms as well as mathematics more broadly.

Students' Understanding of Logarithms

Logarithms are notoriously challenging for students, both in concept and in application. Park and Choi (2013) argued that learner's difficulties with logarithms are often overlooked due to prevalent calculator usage. Indeed, Leopold and Edgar (2008) found that only 21% of university students enrolled in a second-semester introductory chemistry course responded correctly to mathematics questions about logarithms on a standard, calculator-free, university assessment.

Berezovski and Zazkis (2006) proposed interpretive frameworks to explore high school students' understandings of logarithms, based on Confrey's (1991) interpretive frameworks that were designed to explore learners' understandings of exponents and exponential functions. These frameworks were designed to investigate the different conceptualizations and approaches students may use when simplifying expressions or solving problems related to or requiring logarithms. In a sample of 17 students' responses to the task of simplifying $\log_3 54 - \log_3 8 + \log_3 4$, 12 students reached the correct answer, but 9 of those unnecessarily applied the change-of-base formula, suggesting that a calculator was used to reach their final answer. It is therefore suggested that only 3 of 17 students in this sample understood the meaning of logarithms well enough to solve this problem without the aid of a calculator.

There is some evidence to suggest that the correct application of logarithmic identities is challenging for students. Liang and Wood (2005) found that students commonly assumed that $\frac{\log(x)}{\log(y)} = \log\left(\frac{x}{y}\right)$ and that $\log(a+b) = \log(a) + \log(b)$, a finding echoed in the work of Rafi and Retnawati (2018). Liang and Wood (2005) also observed students treating the "log" notation as a sort of "common factor" that could be canceled when it appeared in the numerator and denominator of a fraction, as in the example $\frac{\log(100)}{\log(10)} = \frac{100}{10} = 10$. Indeed, this may suggest that students sometimes do not view logarithms as functions at all or understand the significance of logarithms as functions.

Little other research has investigated students' understandings of logarithmic identities and applications of logarithmic identities in solving equations. Furthermore, in light of the misconceptions outlined above, the accuracy and clarity of the resources students use to support their learning is paramount. Our work begins the process of examining those resources to gain a more complete picture of the information students may consume.

Methods

We performed a preliminary content analysis (Cohen, Manion, & Mason, 2013) on 14 websites. For this introductory study, we chose to focus our attention on websites (as opposed to textbooks or classroom curricula) as these resources are available to all learners regardless of their instructor or the physical resources made available to students. Future studies will explore other sources of content.

To select these websites, we created a Python web scraper that would fetch the top 8 Google results for any given search term(s), as we assumed that most students would select websites from among the top search results for a given search query. We then determined 5 phrases that a student new to the topic of logarithms and logarithmic identities might search to gain a deeper understanding, and we used these as search terms. These search terms were as follows: (a) "Properties of logarithms", (b) "How to use a logarithm", (c) "Logarithms in college algebra", (d) "Logarithmic identities", (e) "Logarithm rules". For this preliminary investigation, we excluded any apps, comment-based websites such as Reddit and StackExchange that lack a systematic content review process, and video resources such as YouTube and KhanAcademy, as we felt that these kinds of resources deserved a more thorough treatment than the preliminary exploration provided in this report. These resources will be explored in more comprehensive future research on this topic. Of the websites left, we chose the top 3 for each search result, yielding 15 websites that we further examined for their treatment of domains of logarithms.

Next, each of the 15 websites was analyzed for the following events: (a) logarithm defined, (b) logarithmic identity stated/defined, and (c) logarithmic identity invoked (e.g., when working through an example). Whenever one of these codes was applied, we then applied one of the following subcodes: (a) domain(s) explicitly defined/stated, (b) domain(s) not mentioned, or (c) only 0 addressed as an exclusion from the domain. To validate the analysis technique of a single author, each author individually coded the same 3 websites, and codes were compared and discussed until 100% agreement was reached on all codes. Note that we only analyzed the specific web pages that were identified by our web scraper. We did not include in our analysis any web pages that were linked to within those web pages.

Table 1. Codes used in our analysis, with possible subcodes corresponding to domain mention.

Codes Examined for Domain			
Codes	Logarithm Defined	Log. Identity Stated	Log. Identity Invoked
Subcodes	Domain explicitly defined	Domain not mentioned	Only 0 explicitly mentioned

Originally, we also looked at occasions where graphs were shown, or extraneous solutions were acknowledged, as these could be considered as implicit mentions of domains of definition for the logarithmic functions involved in a given problem. However, both events occurred very few times (2 each) and these websites did not explicitly discuss these aspects of the problem in relation to function domains. Furthermore, neither of the websites that acknowledged extraneous solutions attributed their emergence to the expansion of a function's domain due to the misapplication of a logarithmic identity; they simply verified that these solutions were extraneous with no discussion of their origins.

After each of the 15 websites was analyzed, we excluded one that defined logarithms once, did not explain the domain on which they are defined, and did not define or use any logarithmic identities. This left us with a total of 14 websites included in our analysis; we present the results of this analysis below.

Results

Once we analyzed the 14 websites in our sample for the above events, we calculated relevant statistics.

Table 2. Coding scheme and instances of occurrence within the 14 websites analyzed

Results			
	Logarithm Defined	Log. Identity Stated	Log. Identity Invoked
Domain mentioned	5	4	0
Domain not mentioned	3	7	11
Only 0 explicitly mentioned	1	0	0
Totals	9	11	11

Most websites from this sample defined a logarithm, and even more stated or invoked logarithmic identities. A small majority (55.55%) of websites that defined a logarithm also stated the entire domain on which logarithms are defined ($x > 0$). One website (11.11%) that defined a logarithm only stated that logarithms are not defined at $x = 0$, but did not indicate that logarithmic functions are undefined for negative x-values. The remaining websites that defined logarithms (33.33%) did not mention any domain.

Although a website in the sample was equally likely to state and invoke a logarithmic identity (78.57%), some websites only had one event or the other. Of all 14 websites, 8 (57.14%) of them both stated and invoked an identity. Given a website that stated a logarithmic identity, it was 36.36% likely the website also stated the domain on which that identity holds. Similarly, given a website that stated a logarithmic identity, it was a 63.64% chance no domain of definition was mentioned.

Finally, given a website invoking a logarithmic identity, there was a 100% chance that the website did not acknowledge any possible changes to function domains that might have been introduced through the application of that identity. This means that, in a website containing an example application of an identity, there was never any discussion about why the identity was (or was not) valid to invoke or how a change in the domain might introduce extraneous solutions or delete solutions. Two websites that invoke a logarithmic identity (18.18%) applied an identity that altered the domains of the functions in the equation. This produced extraneous solutions, which were then discarded. For instance, one website solved the equation $\log(2x + 3) + \log(x) - \log(12) = 0$ using the product and quotient identities for logarithms. This yields one solution, $x = 2$, and one extraneous solution, $x = -3$. The extraneous solution was generated through the application of the product rule to combine $\log(2x + 3) + \log(x)$ into $\log(2x^2 + 3x)$, which changed the domain of the left-hand side of the equation from $(-\frac{3}{2}, \infty)$ to $\left(-\infty, -\frac{3}{2}\right) \cup (0, \infty)$. Websites that did not include examples of extraneous solutions did not mention that they were a possibility. Furthermore, no websites gave examples in which the application of a logarithmic identity caused the deletion of a solution, as in the example we gave in our introduction.

Overall, our data suggest that when a website's goal is simply the definition or introduction of the concept of a logarithm, the domain of the logarithmic function is likely to be included. However, when the focus of the conversation is on logarithmic identities, the domains of the functions involved are unlikely to be mentioned.

Discussion

Our work suggests that websites College Algebra students might use as reference materials when learning about logarithmic identities do not place much importance on the domains of definition of the logarithmic functions used in those identities. It is important to note that logarithmic identities are theorems, and that the satisfaction of the hypothesis of a logarithmic identity requires checking the domains of the functions to be used in that identity. In our earlier example, we can (and perhaps should) restate the power rule for logarithms as follows: "If $\mathrm{m} > 0$, then $\log(m^n) = n\log(m)$". Although we cannot confirm or refute that $x > 0$ in the equation $\log(x^2) = 2$ before we know the solutions, the formal statement of the theorem serves as a warning that, if this equation has negative solutions, applying this identity will cause us to delete them.

As suggested by Berezovski and Zazkis (2006), learners' grasp of logarithms and logarithmic identities is often tenuous. For the hypothetical student who asks, "Why did I get the wrong answer if all my steps were correct?", it may be frustrating to learn that their first step of applying a logarithmic identity led to an error caused by inattention to the domains of the functions involved in the substitution.

In upper-division proof-based courses, a strong emphasis is placed on checking that the hypothesis of a theorem is met before applying the theorem. However, in College Algebra courses, theorems like these are often not even stated as theorems, and students are often instructed to apply them with no regard to their hypotheses. We argue that this sort of "fast and loose" way of doing mathematics at lower levels can lead to confusion at higher levels and reinforces the belief held by some students that mathematics is simply a set of arbitrary rules for moving symbols around on paper.

Future Work

This work is the first step in a program of research that will probe College Algebra students' understandings of logarithms. A key consideration of this research is what information is presented to students in the resources they use to support their learning. Future work will explore the contents of algebra textbooks, comment-based websites, and video resources such as YouTube and KhanAcademy. We will investigate College Algebra students' understandings and perceptions of the applications of logarithmic identities when completing various types of routine exercises given in College Algebra courses. We will also explore instructors' ability to detect these sorts of errors in student work and explore the feedback they provide to their students.

We will also expand this work to attend to other situations in which equality is sometimes claimed by experts or students, but domains are often left implicit, including trigonometric identities and applications of the "factor-cancel" method for simplifying rational functions. We will explore the extent to which domains are acknowledged when using these algebraic and trigonometric tools and how experts and learners react when they encounter unexpected results that arise from their applications.

References

Berezovski, T., & Zazkis, R. (2006). *Logarithms: Snapshots from two tasks* Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague.

Cohen, L., Manion, L., & Morrison, K. (2013). *Research Methods in Education* (7th ed.). Routledge.

Confrey, J. (1991). The concept of exponential functions: A student's perspective. In L. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience* (pp. 124-159). Springer-Verlag.

Leopold, D. G., & Edgar, B. (2008). Degree of mathematics fluency and success in second-semester introductory chemistry. *Journal of Chemical Education, 85*(5), 724-731.

Liang, C. B., & Wood, E. (2005). Working with logarithms: Students' misconceptions and errors. *The Mathematics Educator, 8*(2), 53-70.

Park, E.-J., & Choi, K. (2013). Analysis of student understanding of science concepts including mathematical representations: pH values and the relative differences of pH values. *International Journal of Science and Mathematics Education, 11*, 683-706.

Rafi, I., & Retnawati, H. (2018). *What are the common errors made by students in solving logarithm problems?* The 5th International Conference on Research, Implementation, & Education of Mathematics and Sciences, Yogyakarta, Indonesia.