Learning Approaches in an Introduction to Proofs Course

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We take a situated cognitive perspective to better understand how undergraduate students utilize study resources in their introduction to proofs (ITP) course, particularly compared to their use of these resources in pre-proof courses. Prior research has shown that the transition to proof contains an increase in expectations of rigor compared to previous undergraduate mathematics courses. However, less is known about how students select particular study resources to meet these new expectations. We conducted task-based interviews with four undergraduate students enrolled in an ITP course to better understand the study resources they employed in both pre-proof and proof-based contexts, and the potential motivating factors behind adjustments to study methods. We found that students attempted to migrate previously used resources as well as adopting new resources and were motivated to adjust some study habits due to the novelty of the ITP context, a lack of relevant resources, and instructor modeling.

Keywords: situated cognition, transition to proof, study resources

Studies on the *transition to proof*, or the transition of undergraduate students from pre-proof courses into upper-division undergraduate and graduate proof-based courses, have pointed to difficulties students often face during this transition due to “an expectation of greater student autonomy, a change in the nature of the objects studied, and the seeming unapproachability of professors” (Selden, 2012, p. 39). We hypothesize that this transitory period in students’ mathematics learning could affect the resources they access for studying and/or the ways they use particular resources. In this study, we wish to better understand (a) in what ways student learning approaches may differ between pre-proof and proof-based mathematics courses, and (b) what aspects of their introduction to proof (ITP) course potentially motivate these adjustments.

By *learning approaches*, we refer to both (a) the materials, tools, and/or relationships that individuals use for learning (*resources*), and (b) how individuals use these resources or the actions the individual takes for the purpose of learning (*resource uses*). This construct is similar to that of study habits (Alcock et al., 2020; Hora & Oleson, 2017) in the literature, but with an emphasis on both the resource *and* its use. By doing so, we hope to go beyond broad categorizations for study resources used by students and better understand the nuances of why they choose particular resources to use in their ITP course. Specifically, our guiding research questions are:

(1) What learning approaches do students in an ITP course utilize when presented with proof-based tasks?
(2) When students express differences in learning approaches between pre-proof and proof-based contexts, what motivating factors are associated with these differences?

Understanding what motivates students to adjust their learning approaches during their transition to proof will help researchers and educators better support future students in this transition.

Relevant Literature

Research on students' achievement in ITP courses has found that the transition from pre-proof to proof-based mathematics courses is not seamless for undergraduate students (Moore, 1994; Smith et al., 2017; Stylianides & Stylianides, 2009). Students often must reconceptualize intuitive mathematical understandings into formal definitions (Selden, 2012) and struggle to develop self-regulated (or independent) learning strategies in ITP contexts that allow for personal reflection and improvement of learning strategies when current methods are ineffective (Talbert, 2015). We conjecture that student difficulty in transitioning from pre-proof to proof-based mathematics courses may be exacerbated by students struggling to either adopt new learning approaches or adapt old ones within ITP contexts. This may be aggravated by the diversity of content and approaches in ITP courses (David & Zazkis, 2020) which could make many of the resources students may have previously used, such as standardized mathematics help websites or online tutoring services, less applicable or helpful.

Given the unique and varied nature of ITP courses and the challenges students face in adapting to them, it is valuable to further understand the learning approaches students employ as part of this transition. Robbins et al. (2004) conducted a meta-analysis of higher education studies on psychosocial and study skill factors, finding that college students' academic-related skills impacted their retention and academic goals. In STEM contexts, research on undergraduate students' use of learning approaches is still mostly exploratory, with studies largely working on characterizing students' digital resource use (Aguilar & Puga, 2020; Ogan et al., 2015) or broadly identifying their study habits through self-reports (Alcock et al., 2020; Hora & Oleson, 2017). We intend to contribute to this area by focusing on learning approaches students adapt during their transition to proof, to enable future work in determining which are effective in meeting the unique challenges of ITP.

Theoretical Perspective

We take a situative view of learning in this research project, meaning the environment in which mathematics is learned is inextricably linked to the knowledge that is created (Lave & Wenger, 1991). This perspective acknowledges learning as fundamentally distributed, enacted, and social (Roth & Jornet, 2013). This understanding of cognition leads us to an interest in learning approaches that includes the physical materials referenced and used by students, as well as the actions they take during their learning process and the individuals they work with. In particular, we highlight the following underlying assumptions within this theoretical framework (based on the work of Greeno, 1998): (a) both behavioral skills and conceptual understandings are aspects of participation within a social practice, and (b) changes in the environment (e.g., physical/temporal location, participants present, etc.) lead to distinct conceptual understandings. We use these key assumptions to orient our study design and methodology, as explained below.

Proof-based course environments have unique expectations for the skills and conceptual understandings the students possess, as demonstrated in the literature (Selden, 2012; Stylianides & Stylianides, 2009). This observation, combined with assumptions (a) and (b) from our situated cognitive perspective, leads us to believe that students' learning will change because the situations are unique. That is, since the content and expectations in ITP are distinct from previous math classes, it is reasonable to assume that student approaches to learning will vary within this new context. We assume that since there are some similarities between pre-proof and proof-based courses, including lecture-based instructional style and references to foundational mathematics skills such as algebra and arithmetic, students will initially attempt to adapt their already generated math class "tools" to a new context (Brown et al. 1989).

Methods

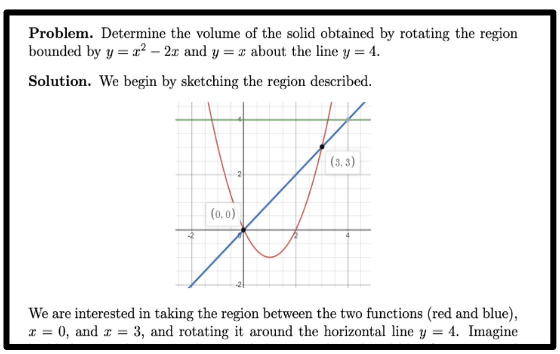
Study Design and Participant Selection

Task-based interviews were conducted with four volunteer students enrolled in a single ITP course at a large public university in the southwestern United States. This research project began as a class project, and the ITP course selection was predetermined for the class project. Students were recruited who had completed a calculus course within the last two years to prioritize recent experiences with the two contexts of interest. The four focal students were interviewed mid-semester, between the first and second summative assessments (Exams 1 and 2).

Informed by our situative perspective on learning, task-based semi-structured interviews (Mejia-Ramos & Weber, 2020) were utilized to explore the learning approaches used by students in pre-proof and ITP courses, as well as motivations for these choices. While our theoretical perspective lends itself to physical classroom observations, practical constraints led us to leverage individual task-based interviews to simulate pre-proof and proof-based classroom environments, an approach with precedent (Roorda et al., 2015; Ticknor, 2012). Ultimately, we assumed that presenting students with pre-proof and proof-based mathematics contexts through these different tasks would encourage students to reflect on different mental processes and potentially unique learning approaches.

Task Design and Data Collection

The interviews contained both contextual and task-based questions. The contextual items included general questions about the participant's goals and experiences in recent math classes (both pre-proof and their ITP course). Four tasks were created, two proof-based and two pre-proof, to simulate the contrasting environments. The tasks were presented as problem statements with worked solutions, as shown in Figures 1 and 2, so the student was not expected to produce an answer to the prompt.

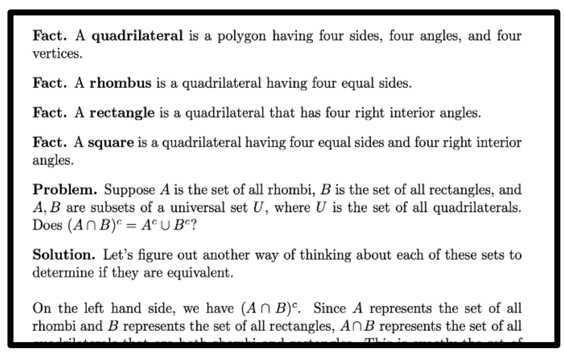


Problem. Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Solution. We begin by sketching the region described.

We are interested in taking the region between the two functions (red and blue), $x = 0$, and $x = 3$, and rotating it around the horizontal line $y = 4$. Imagine

Figure 1. Pre-proof task example.



Fact. A **quadrilateral** is a polygon having four sides, four angles, and four vertices.

Fact. A **rhombus** is a quadrilateral having four equal sides.

Fact. A **rectangle** is a quadrilateral that has four right interior angles.

Fact. A **square** is a quadrilateral having four equal sides and four right interior angles.

Problem. Suppose A is the set of all rhombi, B is the set of all rectangles, and A, B are subsets of a universal set U, where U is the set of all quadrilaterals. Does $(A \cap B)^c = A^c \cup B^c$?

Solution. Let's figure out another way of thinking about each of these sets to determine if they are equivalent.

On the left hand side, we have $(A \cap B)^c$. Since A represents the set of all rhombi and B represents the set of all rectangles, $A \cap B$ represents the set of all

Figure 2. Proof-based task example.

After being presented with each task and given time to review them, the participants were then asked the primary orienting question: *Imagine this was a type of problem you are expected to know how to complete in your class. What study resources might you use to prepare for this type of problem?* Follow-up questions were then asked for each resource mentioned by the student, having students (a) reflect on their use of this resource in their pre-proof and ITP courses; and (b) note any factors that influenced their adoption, adjustment, or removal of learning approaches between contexts. One researcher interviewed the students, while the other researcher(s) present acted as witnesses, keeping memos and supervising the protocol execution for consistency and quality. These interviews were video- and audio-recorded using Zoom conferencing software.

Data Analysis

Thematic analysis (Braun & Clarke, 2012) was used to systematically organize the data set into groups based on students' learning approaches, then compare these groups to identify themes about the factors motivating any changes in these learning approaches. Following Braun and Clarke's six-phase approach, interview recordings were auto-transcribed by Zoom, then manually cleaned and reformatted by a research member. During this process, data familiarization (phase 1) also occurred as the team made memos regarding notable learning approaches. The research team developed initial codes (phase 2) for one transcript across three domains: (a) *Resources*, or specific tools/processes mentioned by the student; (b) *Resource Use*, or ways of implementing a Resource mentioned by the student; and (c) *Motivators*, or explanations given by the student when noting Resources or Resource Uses that changed between pre-proof and proof-based contexts. Resources were generally given shorter descriptions and corresponding tags for easier sorting; Resource Uses and Motivators were generally given longer, low-inference descriptions that might include direct quotes from the text.

After consensus was reached on the initial transcript, the remaining three transcripts were coded by individual team members, adding new tags and revising prior descriptions, which were reconciled and then reviewed through member checking. Each resource tag was re-checked for internal consistency once all transcripts were coded. The research team then developed and reviewed themes (phases 3 & 4) related to the Resource Uses and Motivators mentioned by students. This was a recursive process of moving between the whole dataset and individual coded chunks while discussing patterned or unique findings with the research team. This process led to the development of three themes (phase 5) that illustrate different Motivators mentioned by students for their change in resource use between pre-proof and proof-based contexts. These are further elaborated in the Results below (phase 6).

Results

Results are presented in two parts as relevant to our research questions. First, we discuss the various learning approaches (Resources and their associated Resource Uses) mentioned by students in ITP contexts. Then, we outline the three themes that illustrate common Motivators for change in learning approaches. Student quotes will be used throughout, with the four student names being anonymized with the following pseudonyms: *Aaron, Basil, Chris,* and *Dana*.

Learning Approaches in ITP

Students mentioned a variety of resources and resource uses that they leveraged when presented with proof-based problems. Table 1 below displays a selection of tagged Resources that were mentioned by at least one student while reflecting on a proof-based practice problem, alongside the different Resource Uses students mentioned for these resources in the proof context. Some Resource categories are specific actions taken by students, such as attending instructor Office Hours. Other categories, like Brainstorming and Visual Aids, are more broadly defined, as students would provide different levels of specificity for more general actions, like thinking over a problem statement or visualizing different aspects of the statement. As for the Resource Uses, students would occasionally mention using certain resources simply to find other, more relevant ones. In particular, students often mentioned using the resources of Internet Search and Homework Assignments to find additional practice problems.

Table 1. Selection of resources students mentioned in proof contexts, with associated uses.

Resource	**Description**	**Associated Uses in ITP**
Homework Assignments	Reviewing past (assigned or unassigned) homework problems, including review of instructor comments and/or online homework site feedback	• To find practice problems • To identify patterns in problems
Peers	Working with other students outside of class time, whether online or in-person (or unmentioned)	• To compare answers with others • To "double-check" work with another
Practice Problems	Using practice problems that are similar to the current problem	• To get familiar with similar types of problems • To see a given solution path for a similar problem
Visual Aids	Drawing pictures, diagrams, or other visual aids to represent mathematical ideas	• To visualize geometric objects, function graphs, sets, or set relationships mentioned in problem context
Internet Search	Searching for information about similar content as the problem, with or without specifying an online search engine	• To find similar problems • To review a concept from ITP
Brainstorming	Thinking about or mentally reviewing material; writing down ideas, thoughts, processes, or scratch work; and/or verbalizing ideas to self before formally writing down solution steps	• To prepare oneself and take in the information

We also note that these are not all of the Resources mentioned by the students. Some resources mentioned in proof contexts, like Online Messaging and Question Patterns, were cut for space. Students also mentioned using some resources, including Computational Devices and Online Text Material, only in pre-proof contexts. This list is not meant to be an exhaustive list of resources used by students in ITP courses, but rather a starting point, developed from interviews with four students.

Motivating Factors for Changes in Learning Approaches

Three themes were developed to highlight consistent factors that students mentioned as Motivators for change in their Resource Use between pre-proof and proof-based contexts: *Novelty of Proof Content*, *Relevance of Resources*, and *Instructor Modeling & Course Structure*.

Theme 1: Novelty of proof content. Students mentioned changing their learning approaches in light of the novelty of the new content in their ITP course. Generally, participants noted a difference in how problems were structured in their course compared to pre-proof courses. As stated by Dana, "[ITP course] isn't really, you know, solving problems. It's more just finding relationships between things […] they're just completely different kinds of problems." These differences were often characterized as an increased level of complexity or abstraction in the problems. In response to these differences, some students would alter their Resource Use while retaining the Resource itself between pre-proof and proof contexts. For instance, Chris noted that due to concepts in ITP being "fairly abstract," he had increased use of office hours compared to pre-proof courses. Similarly, Basil noted a decrease in his use of online materials, stating that "you can't check a proof like that online [*laughs*] that's not, that's not how that works."

The most common motivation for change due to this new math context was the ability for the same problem to be answered sufficiently with different approaches. All four participants mentioned this as a motivation for change. Dana, Chris, and Aaron cited increased difficulty in using the Resource of Question Patterns, noting that "[ITP] is just harder, because, like, it's more than two patterns for more questions," and "some things I can't [do based on memory], just because again, there's not always one correct answer for everything." Dana particularly mentioned that this decreased the effectiveness of the Practice Problems Resource for her because the variety of solutions made it so she was unable to check her work. In sum, the novelty of the mathematical context in ITP, in terms of both the complexity of questions and multiplicity of solution paths, was related to students' adjustment of their learning approaches.

Theme 2: Relevance of resources. The second theme highlights how some students utilized learning approaches based on the perceived relevance of these approaches to their learning needs. For instance, one student did not use office hours in their pre-proof course, because all lectures were recorded, making additional in-person sessions unnecessary. In cases where students would try to implement a previously used learning approach in ITP, multiple participants commented on struggling to find resources that matched the nuances of their proof problem statements. Basil shared his reluctance to use internet searches, stating, "I would not look up on the internet for this problem, 'cause there'd be way too many miscellaneous things that I would not want to look at. I'd get some third-grade geometry stuff that would show up." This illustrates a common challenge mentioned by students of sifting through irrelevant material, which can overwhelm students with content that doesn't directly help them achieve their learning goals.

Students conjectured that this irrelevancy of material was related to the diverse and inconsistent ways ITP courses are taught. Aaron expressed,

> I think the material is not maybe taught exactly the same between different colleges and even different classes in every college. I feel like each professor's ideas about proofs and kind of the important groups to teach are different […] I think it would be hard to find specific videos on exactly what we were learning.

This mirrors what David & Zazkis (2020) found when noting the various ways ITP courses are implemented in the U.S. The wide variety of online resources available for pre-proof courses contrasts with the specificity required in proof-based learning, where students struggled to find resources that they found relevant to their ITP course content.

Theme 3: Instructor modeling & course structure. Finally, we noted that change was sometimes motivated by the format of the ITP course itself. The *Instructor Modeling* portion reflects moments when students explicitly mentioned adopting a resource after seeing their instructor using or advocating for the resources during instruction in their ITP class. Aaron and Dana mentioned instructor modeling as a reason for their adoption of certain resources in ITP. When it came to Brainstorming, Aaron noted, "our professor makes us think a lot before we talk to our group mates. She's like, ‘look at it for like 3 or 4 min.’ So, then I just do that. And then I guess that kind of just stuck.” Aaron mentioned not using this resource previously in calculus contexts, instead choosing to “start writing stuff down almost immediately.” Similarly, Aaron mentioned using set diagrams and drawings of geometric shapes in ITP explicitly because the instructor used that resource in class while not finding a need for visual means in calculus. Dana had a similar relationship to the Visual Aids Resource, finding that set diagrams were a great way to visualize material in ITP, despite preferring equations to graphs in calculus contexts.

Generally, this theme represents the recurring instances where students mentioned the overall structure and content of the course (and the implementation of that content) as a motivator for resource adoption in their ITP course. One such motivator was the allotment of class time to use particular resources. Chris used the Peers resource in ITP but not in calculus contexts, noting that, “I think [ITP course] is special, because we had peer discussion. We, I also wanted to, like, note down my thought process during it. But in Calculus 2, [it] is pretty much like taking notes from the professor.” Because Chris was explicitly given the time to talk with peers in their course, he utilized that resource. Other such motivators include being given additional unassigned homework problems that could serve as Practice Problems, accessing course slides that served as a notetaking guide for Class Notes, and leveraging handouts given as Course Materials. Though these aren’t motivators specific to ITP courses, they seem to indicate a necessary condition for the use of some resources: to be allowed access and permission to use said resource in the course.

Discussion

Overall, the themes noted in these students’ motivators for changes in learning approaches paint a picture of the transition to proof as being one of uncertainty and novelty, echoing what has been previously found about the challenges of the transition to proof (Selden, 2012; Stylianides & Stylianides, 2009). The students reflected on using a variety of learning approaches in their ITP course, both adapting resources from pre-proof math classes and adopting new resources. They also commented on the factors that motivated some of these changes in learning approaches. We noted the novelty of the material leading to difficulty and struggle, the resources lacking relevance to their course context, and a reliance on instructor modeling for effectively studying in the ITP course as themes in motivation. The overall student discomfort in the midst of transition is exemplified by Chris, who said, “Sometimes [in ITP course], like, the concept is fairly abstract. So, I would have to, like, go to office hours to ask for, like, more detail [...] but in calculus, everything that I learned is pretty much, like, computable. Like, you can calculate it easily. So, it's more thorough.” While this is a single example, the comments and tone of the students we interviewed indicated a general discomfort and feeling of uncertainty, especially noting the loss of calculation, a solving technique they felt comfortable with.

This research project serves as a starting point for better understanding the types of resources used by students in ITP courses, particularly compared to previous learning approaches they applied in pre-proof contexts. Future work may expand on this study by increasing the number of participants or leveraging the list of student-mentioned learning approaches to run a broader survey-based study asking students to reflect on their use of some of the mentioned learning approaches. In light of our last theme, if instructor modeling and course structure are motivating to adopt new learning approaches, these components could be used intentionally to foster effective approaches and support students as they are introduced to the novel and abstract world of proof.

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