Here's What "I" Do: Investigating the Actions Students Take in Proof Comprehension Tasks

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Literature around the reading of proof has focused on experts and the strategies they use to understand the texts they are reading (Fang & Chapman, 2020; Paul, 2018) or in comparing students to experts in what the reader focuses on as they read using eye-tracking technology (e.g., Inglis & Alcock, 2012). Few have investigated the students and their strategies (Weber, 2015). In this preliminary report, we describe the most frequently seen actions taken by participants during various proof comprehension tasks. Our findings indicate that students use a variety of possible strategies to make sense of proofs they are reading. Some of these possible strategies are found to be used by mathematicians and secondary mathematics teachers, while other possible strategies are found to be used in general reading practices.

Keywords: Proof Comprehension, Reading Proof, Student Perspective

Investigations into what students do as they attempt to understand proof can provide insight into how students think about proofs, such as what it means to understand a proof, as well as identifying what strategies students commonly rely on when they arrive at a confusing part of a proof. Both math education and literacy scholars have investigated various ways in which mathematicians read proof (e.g., Fang & Chapman, 2020; Inglis & Alcock, 2012; Paul, 2018; Weber & Mejia-Ramos, 2011), though few have investigated how undergraduate students read proof (Inglis & Alcock, 2012). In proof comprehension, scholars focus on assessing or improving students' comprehension of proof (e.g., Hodds et al., 2014; Mejia-Ramos et al., 2017; Roy et al., 2017), with less attention on what the students do as they read proof for understanding (Dawkins & Zazkis, 2021; Weber, 2015). Considering the student and the actions they take can provide important information in the development of interventions. These interventions can be aimed at further developing these skills by building on student starting points. In this preliminary report, we set out to answer the following research question: What actions do students take to help with their understanding of a given proof during proof comprehension tasks?

Literature Review

Scholars have investigated various ways in which mathematicians and students read mathematical text. These could be through comparing what experts and novices attend to while they read proof using eye tracking for proof validation tasks (Inglis & Alcock, 2012), how mathematicians determine the validity of a proof (Weber, 2008), why mathematicians read proofs (Mejia-Ramos & Weber, 2014; Weber & Mejia-Ramos, 2011), or in what mathematicians do as they read a proof about unfamiliar material (Wilkerson-Jerde & Wilensky, 2011).

Literacy experts have also investigated the reading habits of mathematicians and secondary education teachers. Fang and Chapman (2020) discuss a case study of the reading strategies employed by one mathematician as they read a journal article. Paul (2018) describes the different reading strategies used by various secondary education content teachers, one of which is a mathematics teacher. They both report various strategies such as annotating, evaluating and verifying, corroboration (such as using prior knowledge or other texts), and strategic reading (such as rereading, sectioned reading, or skimming).

The studies described above focus on comparing mathematicians to students or focus purely on experts in how they read and the strategies they may employ as reading. Little has been conducted to see what students do as they read mathematical text. One exception is Weber (2015) who investigated what strategies students use as they read proofs. Weber (2015) and Samkoff and Weber (2015) report seven strategies used by the students or recommended to be used by mathematicians. Mathematicians recommended readers should make sure they understand the definitions of all terms (Samkoff & Weber, 2015), while students were reported to see if the proof can be broken into independent parts, identify what proof methods are being used, and see why confusing statements are true with a particular example (Weber, 2015).

Theoretical Framing

Afflerbach et al. (2008) defines reading strategies as "deliberate, goal-directed attempts to control and modify the reader's efforts to decode text, understand words, and construct meanings of text." (pg. 368) Within the realm of reading proofs, we define a strategy as deliberate acts from the reader with the goal of understanding some subset of a given proof. Thus, we define a proof comprehension strategy as any action of the reader that is done outside of the proof's text with the goal to understand some portion of a given proof (inclusive of the whole proof). By outside of the proof's text, we mean the student is engaging in furthering their understanding through actions that are not done or recommended within the proof. In this preliminary report, we focus on the actions the students take that may make up a proof comprehension strategy.

Methods

The study described here is part of a larger dissertation study investigating the strategies students use to understand proof. In this section, we describe the methods pertinent to the results. Eight participants were recruited for this study, each of which had completed an introduction to proof (ITP) course the semester prior to data collection. Table 1 shows participant information using participant chosen pseudonyms.

Table 1: Participant Information

Participant	Race/Ethnicity	Gender	Grade Received in ITP
Adrian	White or Caucasian	Non-binary	B
Claire	White or Caucasian	Female	B
Erik	White or Caucasian	Transman	A
KTRM	Latin/Hispanic	Female	C
Marie	Latin/Hispanic	Female	A
Olivia	Mixed Race	Female	B
Tifanni	Latin/Hispanic	Female	A
Zeus	Latin/Hispanic	Male	B

Each participant was met with for seven semi-structured interviews. Four of those interviews involved a proof-comprehension task where participants were asked to read a provided complete and correct proof and to think-aloud as they tried to understand the proof to the best of their ability. Table 2 shows the four theorems for context of the study (proofs have been excluded due to space). After finishing the task, participants were asked various questions on their beliefs of their understanding of the provided proof, the difficult parts of the proof, and about the actions taken as seen by the interviewer.

Interviews were video and audio recorded. Video recordings were open coded (Strauss & Corbin, 1998). Instances were first coded by the first author based on ostensive cues (verbal, written, or gestural indicators). Through the initial open coding, code names and descriptions were given to each instance. Once all instances were open coded, a second round of coding was done by the first author to refine code names (actions), and subcodes were also given during this time to further differentiate the actions taking place.

Table 2: Theorem Statements for Proof Comprehension Tasks

Theorem	Theorem Statement
1	If $a, b \in \mathbb{N}$, then there exists integers k and l for which $gcd(a, b) = ak + bl$.
2	Suppose R is an equivalence relation on the set A. Then the set $\{[a] \mid a \in A\}$ of equivalence classes of R forms a partition of A.
3	Let $f: A \rightarrow B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. Furthermore, if f is bijective, then f^{-1} is also bijective.
4	The sets $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality.

Results

Through open coding of the instances, 34 different actions were found amongst the 8 participants across all four proofs. Due to limited space, we focus our results to the most frequently used actions. To determine this, we set a lower bound of 5% of the total final actions coded (1096) and required the actions to be done by at least five out of the eight participants. Seven actions were found to meet these requirements: Asks for Meaning, Creating Warrant, Identifying Proof Structure, Making Connection, Marking Proof, Rereading, and Writing Information. Table 3 shows the frequency of each action per participant, with each student's highest frequency bolded.

Table 3: Action Frequency per Participant

Student	Coded Action						
	Asks for Meaning	Creating Warrant	Identifying Proof Structure	Making Connection	Marking Proof	Rereading	Writing Information
Adrian	11	**39**	9	13	0	14	9
Claire	6	**36**	5	13	1	11	25
Erik	5	**20**	18	6	4	5	19
KTRM	4	**15**	0	4	2	6	7
Marie	5	**22**	14	15	9	14	17
Olivia	11	15	3	6	11	16	**25**
Tifanni	7	11	8	8	10	**32**	8
Zeus	10	12	16	4	**41**	9	7
Total	59	170	73	69	78	107	116

The creating warrant action was the most frequent action taken by all but three participants (Olivia, Tifanni, and Zeus). In these instances, participants focused on justifying an explicit or implicit claim made in the proof. The justifications presented by the participants were coded even if the justification was incorrect or incomplete (such as deciding to move on from the claim without finishing their justification). The data used to warrant a claim varied amongst instances, such as filling in skipped algebraic steps, using a previously self-made conjecture, using the definition of a concept, using a self-made diagram, using a self-made example, using their own reasoning, using a previous part of the proof, or using their own previously made written work.

The most frequent action for Olivia was writing information, which was among the top seven actions for all remaining participants (except Adrian). This action consisted of a participant writing information either on a separate sheet of paper or on the sheet of paper containing the proof. This action was coded specifically when a participant wrote information that was already presented to them (such as a definition that was given to them by the interviewer on a separate slip of paper, the defining of a set or function in the proof, a claim made within the proof, or the rewriting of an example they created) or when the participant wrote a conjecture of their own. Further variety was seen through the reasoning behind the writing of the information. Participants commonly stated they wrote information for two reasons, either to keep track of important information for ease of referencing or to help them understand information presented to them. Figure 1 shows the notes Olivia took during their time reading Theorem 4's proof. In these notes, Olivia wrote the function f that is used, initially doing this to keep track of the function and have an easier time referencing it. She later used this written work to make sense of the function in what values n and a_n could be.

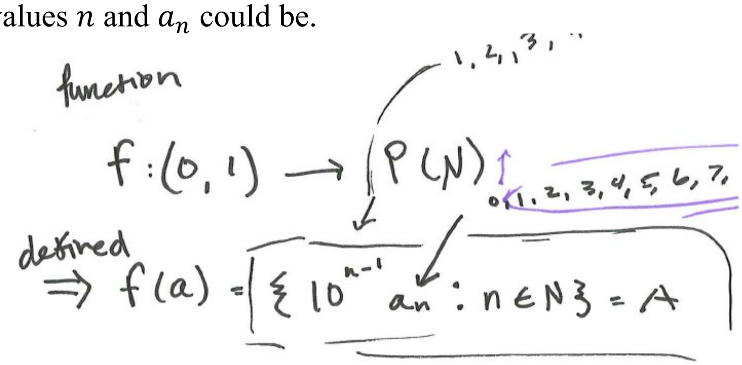

Figure 1: Some of Olivia's Notes for the Proof of Theorem 4

The most frequent action for Tifanni was rereading, which was among the top seven actions for all participants. These instances varied in what was being reread, such as rereading some part of the proof or a provided definition. If rereading some part of the proof, participants could have reread the whole proof (including or excluding the theorem statement) or a line, paragraph, or theorem of a proof. Participants would at times also have specific styles of rereading. Two such styles were rereading but stating the highlights, as if creating a summary of the proof (categorized as a subcode), or purposefully rephrasing (subcode) a line of the proof for ease of reading. For example, as Tifanni's read the proof to Theorem 3, she reads the line "Let $b \in B$" as follows: "Let bee exist in bee, oy [sic], let little b exist in big b." Here Tifanni is making a distinction in the letters noting "Which b am I talking about? Like in my head. Because of course I'm seeing them, but since I'm reading it, I'm like, so I'm talking about *this b*, not *this other b*" (emphasis added). Participants frequently reread to better understand what they just read, either immediately or after having established more information from later text or a definition.

Marking the proof was the most frequent action for Zeus and within the top seven for Marie, Tifanni, and Olivia. This action commonly consisted of participants underlining, dotting, or boxing parts of the proof. This was typically done to draw attention to key claims made in the proof – such as in Theorem 3's proof with the final claim of an argument being that f is onto. Participants would also use these markings to highlight parts of the proof they were still confused about. Erik, Marie, Tifanni, and Zeus had identifying proof structure in their top seven most used actions. Identifying proof structure actions varied between what the participants were identifying. These ranged from identifying independent parts of the proof, what the proof needed to show to prove the theorem, what proof method was being used, or what purpose a part of the proof was playing towards proving the theorem.

Making connections was in the top seven most frequently taken actions by all participants except for Olivia and Zeus. When making connections, participants would identify a relationship (typically through similarities and differences) – or lack thereof – between various texts or concepts. Participants commonly discuss a noticing between the proof and some other text – such as an example the participant created, another part of the proof, the theorem statement, or a definition of a concept. For example, Theorem 2's proof uses a contrapositive argument to show that equivalence classes are disjoint. Participants frequently discussed the differences between part of the definition of partition (that if two sets are not equal they are disjoint) and the start of one of the paragraphs of the proof (assuming two equivalence classes are not disjoint).

Lastly, asking for meanings was in the seven most frequently taken actions by all participants except Claire, Erik, and Marie. Participants would ask the interviewer for the meaning of a definition, notation, or theorem referenced in the proofs. Participants commonly discussed needing the definition to remind themselves of a concept or to help make connections between the proof and the definition. For example, participants frequently stated they needed to use the definition of partition to help see the structure of Theorem 2's proof, as they commonly could not recall (or were unfamiliar with) the definition of partition.

Discussion

Participants took a variety of actions as they attempted to understand the four proofs they were shown. Similar to Weber (2015), participants created warrants for claims, identified different proof structures, or attended to the meaning of terms used within the proof. Though, in our study we see more distinctions in the warranting of claims and expand on these distinctions. First, our participants warranted claims in a variety of ways. Either through examples, diagrams, definitions, their own reasoning, or using prior parts of the proof. This shows students are using more than just examples to make sense of claims they do not see immediately. Additionally, participants not only attended to the meaning of terms within the theorem statement (as described by Samkoff & Weber, 2015), but this extended into the proof itself. Participants frequently ask for definitions they were able to (and did) recall, as a way to ensure their thinking was correct. We also identified actions novel to Weber (2015), such as rereading, marking proof, writing information, and making connections.

Though novel to Weber's study on undergraduates, these actions can be seen being used by mathematicians and secondary mathematics teachers in literacy studies. Fang and Chapman (2020) describe their mathematician participant corroborating (making connections), rereading, as well as evaluating (creating warrants). Paul (2018) describes secondary mathematics teachers rereading, annotating (writing information and marking proof), and corroborating (making connections). Our study shows that students are doing actions similar to those found in the literature by taking general reading strategies and using them while reading proof. That is, novice provers bring a rich array of strategies for comprehending proofs. By identifying the strategies students bring into the proof setting, we are taking the first step in developing a theory of what strategies students use, what contexts lead to the use of certain strategies, and when and how particular strategies serve to be productive for students to generate insights. This can lead to targeted instruction and support for students to develop their emerging proof comprehension strategies. Future research from the larger dissertation study will be to describe these strategies in what kinds of strategies students use, how they are used, and how they may differ between uses. Limitations to this study are that participants saw a finite set of proofs and the actions they took may have been due to the types of proofs or in their familiarity with the content as they had just completed an introduction to proof course.

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