Mathematical Induction: Which Written Proofs or Visual Proofs are More Convincing?

Mario Abel Gonzalez
Texas State University

This study uses a survey to investigate mathematical convictions in tournament-style comparative judgements for written and visual proofs. The survey uses four theorems that can be proven using mathematical induction with each having four proofs: two written and visual. The visual proofs were equally convincing, and the respondents indicated that they believed these visual proofs to be mathematical proofs but not prove their theorems for all values n. The written proofs by mathematical induction were often the most convincing proofs—winning the overall tournament. Familiarity and content seemed to be prevailing factors for judging whether one proof was more convincing than another.

Keywords: Mathematical induction, comparative studies, visual proofs, mathematical conviction

Mathematical conviction (Brown, 2018), the ways in which one is persuaded about the truth of a theorem given a proof, has been deemed a nebulous concept posing challenges to empirical study. This study seeks to gain new insights by exploring how students respond to two types of proofs with specific features: visual proofs and written proofs by induction. Though visual proofs have a contested status as proofs (Brown, 2008), they may offer useful tools for investigating what types of proofs may be convincing (Alcock & Simpson, 2004; Relaford-Doyle & Núñez, 2021). Mathematical induction (MI) has been well-documented as being difficult for students to understand (Ernest, 1984) where issues may arise at any step (Stylianides et al., 2016), but it may also provide interesting insight about when and how students are convinced by it (Brown, 2014; Stylianides, 2007). The purpose of this study is to investigate how these two features influence student proof convictions using comparative judgments (c.f., Brown, 2018).

Background Literature

Most literature focuses on whether students are convinced by proofs or something other than proof (Stylianides et al., 2017). In this study, I am concerned with the features of proof that make them more convincing in comparative judgement tasks.

Components for this Study

There are a variety of studies on mathematical convictions which report a multitude of factors influencing whether and how a mathematical proof is convincing. Common aspects influencing this judgement are familiarity with the content and method (Brown, 2018; Chazan, 1993; Healy & Hoyles, 2000), example-use (Healy & Hoyles, 2000; Weber et al., 2022), and whether one understands that proof (Brown, 2018; Selden & Selden, 2003).

Visual elements may also influence mathematical convictions because those visual elements may convey the same features that written proofs can (Relaford-Doyle & Núñez, 2021; Zhen et al., 2016). This can be done perceptually (Harel & Sowder, 1998; Miller et al., 2022) and might be salient enough for students to gain insight into the argument (Weber, 2010). There is also evidence that visualizations and convincingness interact (Alcock & Simpson, 2004, 2005). Combining recursion and visual elements can also be helpful for becoming convinced by a mathematical argument (Knuth, 2002).

Recursion can provide an entry point into MI (Harel & Sowder, 1998). MI is susceptible to cognitive gaps (Norton et al., 2022) which may be alleviated through example-use (Stylianides et al., 2016). So, combining visual aspects, in the form of proof, and mathematical induction provides the foundation for this study's methods because MI can be formulated to reveal recursive relationships thereby influencing mathematical convictions.

Methods

I sampled from a discrete mathematics 2 course at a large southern university in the United States using a survey which received 15 responses. For four theorems, four proofs were created or adapted from prior literature. Two of these were visual proofs, one that demonstrated how different cases were related (visual induction-promoting proofs, VIPPs), and the other showed a single case to be taken as general, often when $n = 7$ (visual general-promoting proofs, VGPPs). The other two proofs were written proofs: an MI proof (WMIPs) and a non-MI proof (WNMIPs).

Instruments

The survey features a tournament-style comparative judgement design with four theorems with each of these having the previously mentioned types of proofs. Every written proof avoided prompting phrases for proof methods such as "this is a contradiction" or "by the principle of mathematical induction." Figure 1 below shows the general structure for the tournament.

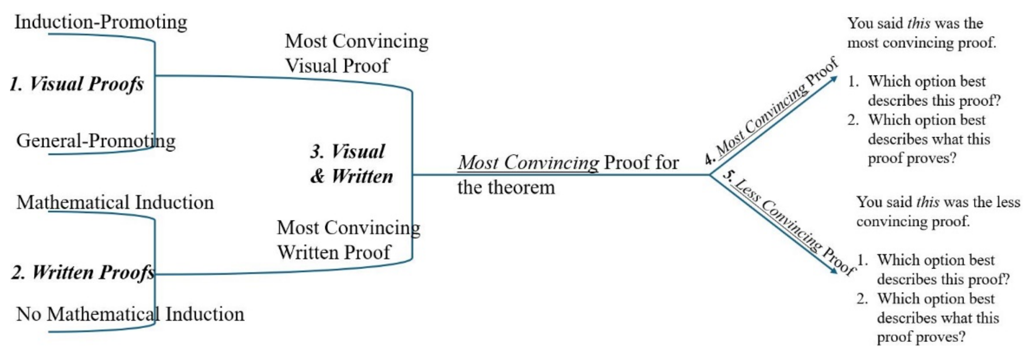


Figure 1. The General Structure of the Survey for the Four Theorems.

In Figure 1, the numbers indicate the order presented within the survey. Visual proofs were always compared first, written proofs second, and then a visual and a written proof third. The order of all answer choices was randomized to prevent a priming effect. The exact phrasing for prompting students to select the more convincing proof was "which proof, in your opinion, is the most convincing? In other words, which proof better persuades you of the truth of the theorem?" (Brown, 2018, p. 8). Students were allowed to select an option that stated both proofs were equally convincing. After the third bracket, the students were shown their most and least convincing proofs and asked what type of proof they were and what they proved. Borrowed from Brown (2018) and Relaford-Doyle and Núñez (2021), these questions confirmed student engagement and how they interpreted the proofs (the type and how general).

The first and second theorem (see Table 1) were intended to be familiar and were often used as either examples or homework problems in ITP courses. The third and fourth theorems were intended to be unfamiliar. That is, these theorems were likely not seen as homework problems or examples in class. All WNMIPs were direct proofs except the one for Theorem 1 (see Brown, 2018, p. 11, Argument 1B). Table 1 presents the four proofs for one of the four theorems.

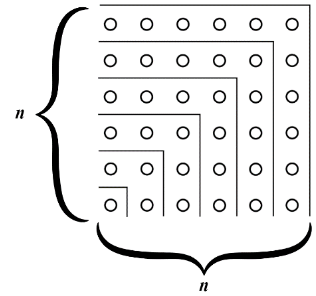
Table 1. Theorem 2 and its Proofs

Theorem 2: Let n be a positive integer, then $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

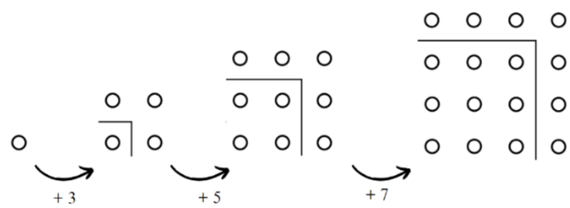
Proof 2A (WNMIP)

$$\begin{aligned} 1+3+5+\cdots+(2n-1) &= \sum_{k=1}^{n}(2k-1) \\ &= 2\sum_{k=1}^{n}(k) - \sum_{k=1}^{n}(1) \\ &= 2\frac{n(n+1)}{2} - n \\ &= n^2 \end{aligned}$$

Proof 2B (VGPP)[a]



Proof 2C (VIPP)[a]



Proof 2D (WMIP)

Let $n = 1$, then $2(1) - 1 = 1^2$.
Suppose that the theorem is true for some positive integer k.
It follows that,
$1 + 3 + 5 + \cdots + (2k - 1) = k^2$.
Thus,
$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$.
It follows that for all natural numbers n,
$1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

[a] Adapted from Brown (2008, p. 35)

Results and Analyses

There were several preliminary analyses performed. Table 2 was created to specifically see how convincing a type of written and visual proof were for theorems that could be proven with MI. Table 3 was created to follow students' decisions and which proof type was the most convincing. Table 4 was created to investigate how students understood the visual proofs. The current sample size cautions against inferring too much from these frequencies, but I illustrate the analysis to be performed on the larger sample set being gathered.

First and Second Bracket Results

The first and second brackets compared two visual proofs and two written proofs, respectively. These results are summarized in Table 2.

Table 2. First and Second Bracket Results Across the Four Theorems

	First Bracket		Second Bracket		
	VIPP (15)	VGPP (15)	WMIP (15)	WNMIP (15)	Total (30)
Theorem 1	7	5	11	4	27
Theorem 2	6	9	8	6	29
Theorem 3	5	9	11	3	28
Theorem 4	12	3	11	2	28
Total	30	26	41	15	112

The numbers appearing next to the headers indicate the maximum number possible for an entry in that column. The entries in the columns do not always add to 15 because some students said both proofs were equally convincing. In the first bracket, the visual proofs were nearly chosen as often as each other. Theorems 1 and 4 had higher instances of the VGPP chosen as more convincing than the VIPPs whereas Theorems 2 and 3 were the other way around. In the third bracket, the WMIPs were chosen nearly three times as often as the WNMIPs with those proofs being chosen almost as frequently as each other for Theorem 2. These findings suggest that students had preferences regarding which visual proofs were more convincing, but they did not always align to the distinction between VIPP and VGPP across the tasks. By contrast, students found the MI proofs more convincing, at least within this limited sample.

Overall Tournament Results

The students made a total of 180 selections across the four theorems. These results are summarized in Table 3.

Table 3. Overall Tournament Results Across the Four Theorems

		Student															
		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	Total
Third Bracket Only (4)	VIPP	0	0	3	0	1	2	0	0	0	0	0	1	3	3	2	13
	VGPP	0	1	0	0	0	2	0	2	0	1	0	0	1	1	1	9
	WMIP	3	1	1	3	1	0	2	1	1	3	3	2	0	0	1	23
	WNMIP	1	0	0	1	0	0	2	0	1	0	0	1	0	0	0	6
Totals Across All Brackets (12)	WMIP or VIPP	8	6	10	9	8	7	5	5	5	8	7	7	6	7	9	107
	WNMIP or VGPP	4	4	2	3	0	5	6	6	5	4	3	2	4	5	3	56

The numbers appearing next to headers indicate the maximum number possible for that row. In the third bracket, students more often selected the WMIPs as the winner of the tournament. This is interpreted as WMIP being the most convincing proof for these 15 students across the four theorems regardless of its content. The last two rows were used to help determine whether the WMIPs or VIPPs were in fact more convincing, in general, than their counterparts. The results showed that, when given the chance, the students chose those proofs as the more convincing nearly twice as much. This also shows that 10 of the 15 students chose the WMIPs or the VIPPs more than half the time while only one student chose the WNMIPs or VGPPs more than half the time.

Students' Descriptions of the Proofs

One component of this study was to determine how students viewed the visual proofs. The fourth and fifth components of the survey structure were used for this (see Figure 1).

Table 4. Students' Descriptions of the Proofs

Question Asked	Option on Survey	VIPPs (34)	VGPPs (26)	WMIPs (45)	WNMIPs (15)
Which option best describes this proof?	Direct Proof	14	9	7	5
	Contrapositive Proof	2	1	1	2
	Proof by Contradiction	1	0	1	2
	Proof by MI	8	7	32	6
	This is not a proof.	2	4	1	0
	Something else.	4	4	2	0
	All values of *n*.	19	17	37	12

Which option best describes what this proof proves?	Specific values of *n*.	10	8	5	3
	Something else.	2	2	2	0

In Table 4, the numbers in the parentheses next to the headers indicate the number of times that type of proof entered the third bracket therefore entering the fourth or fifth component of the survey. The two most common descriptions for the visual proofs were direct and MI proofs. A few students stated that the visual proofs were not proofs or described them as something else. These latter responses consisted of either uncertainty about the status of the proofs and picture or visual proofs. Notably, the number of responses indicating the visual proofs are mathematical proofs and those indicating that the proof proved the theorem for all values of *n* do not match. This was interpreted as students being convinced by something that they identified as mathematical proof while providing evidence that it did not meet the generality of a proof.

Discussion and Conclusion

In the first bracket, students chose the visual proofs with almost the same frequency. So, it cannot be yet determined whether one type is more convincing than the other. However, Theorems 1 and 4 had higher instances of VIPP while Theorems 2 and 3 had higher instances of VGPP. On one hand, the students may have "seen" induction (Brown, 2008) in the VIPPs for Theorems 1 and 4 but not Theorems 2 and 3. On the other hand, Relaford-Doyle and Núñez (2021) noted students might understand and reason about a diagram purporting to promote mathematical induction in ways that separate the diagram and their knowledge of mathematical induction. So, the result might be that the diagram can be helpful or convincing but for reasons unrelated to mathematical induction or other formalized methods of proof. Nevertheless, initial findings suggest that students have preferences based on some features of these visual proofs.

In the second bracket, the WMIPs were most often chosen as the more convincing proof. This can be interpreted as students' instruction on MI heavily influencing their choices on matters of conviction. That is, it is easily identifiable and familiar (Brown, 2018). The students' propensity for choosing the WMIP as the winner of the tournament can again be interpreted as students' familiarity with this proof method as a deciding factor for the second and third brackets. Notably, the WNMIPs for Theorems 2, 3, and 4 were not so often chosen. These proofs featured almost exclusively summations, manipulations, and few words. This might suggest that the idea that students value familiarity sanctioned in their classrooms despite prior research reporting its technicalities and difficulty (Ernest, 1984; Stylianides, 2007). In addition to this, the WMIPs and VIPPs were also chosen almost twice as often as WNMIPs or VGPPs throughout the survey. However, WMIPs constituted almost 60% of these selections. This again lends to the idea that the familiarity of written proofs by MI enhanced their convincingness.

For how students viewed the visual proofs, most students either viewed them as direct proofs or proofs by MI. However, not all those who selected an explicit proof type (50 instances) also said the proof proved the theorem for all values of *n* (36 instances). This is interpreted as those students possibly holding varying definitions or conceptions about mathematical proof (Morris, 2002, 2007; Recio & Godino, 2001). In other words, students may be convinced by something other than what many communities call "mathematical proof" even in more advanced mathematics courses such as discrete mathematics 2 Because these results and analyses had a limited sample set, I am gathering a larger data set this Fall.

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