Undergraduate Students' Conceptualization of Repeating Decimals

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The mathematics education community has examined how students conceptualize repeating decimals. Yet, there is potential for further investigation of how undergraduate students conceptualize the addition of repeating decimals. This paper explores pre-service teachers' conceptualization of repeating decimals. Each participant answered three open-ended questions, and our data consisted of these written responses. Our analysis highlighted that the school conventions hindered many participants from seeing how to add repeating decimals without truncating them. Initially, most participants viewed repeating decimals as processes that approximate rational numbers. Many participants came close to considering repeating decimals as mental objects rather than processes. Our data indicates that asking participants how they would check if they had correctly added repeating decimals enabled them to overcome epistemological obstacles that hindered them from seeing repeating decimals as numbers.

Keywords: APOS theory, Geometric series, Infinite processes, Infinite series, Repeating decimals

Infinite processes are vital in mathematics education, especially in undergraduate mathematics courses. Common topics that involve the notion of infinity are sets, sequences, and limits. Most students first encounter infinite series in calculus courses but rarely explicitly in school mathematics. Hence, mathematics education researchers know much more about how students think about sets, sequences, and limits (Dubinsky et al., 2005a; Dubinsky et al., 2005b) but have yet to study how students conceptualize series extensively. According to Kung and Speer (2013), the mathematics education community still needs to amass "an extensive research base that provides insights into student understanding of this topic [of infinite series]" (p. 420) because we know very little about "how students think about this topic" (p. 420).

Repeating decimals exhibit a duality between potential and actual infinity that may appear paradoxical. On the one hand, a repeating decimal exemplifies potential infinity when one views it as a continuous process of forming digits to express a rational number through long division. On the other hand, a repeating decimal is an instance of actual infinity because it represents a fixed number. This paradoxical duality between potential and actual infinity baffles many students. Some authors (see, for example, Dubinsky et al. (2013), Edwards and Ward (2004), Sierpińska (1987), Sinclair et al. (2006), Weller et al. (2009, 2011), and Yopp et al. (2011)) have investigated how students think about repeating decimals and their relationship to infinity and limits. However, researchers have yet to examine how students conceptualize arithmetic with repeating decimals. Weller et al. (2009, 2011), Yopp et al. (2011), and Dubinsky et al. (2013) have found that many pre-service teachers struggled with repeating decimals and did not believe that $0.\overline{9} = 1$. Sinclair et al. (2006) developed an instructional treatment that "draws attention away from the procedure of conversion" to allow learners to view representations of fractions in decimal form "as objects rather than processes" (p. 184). They hypothesize that pre-service teachers view the standard conversion of decimal numbers into fractions as a process "that allows them to turn decimal numbers into fractions rather than as a general relationship between fractions and repeating decimals" (p. 189). Sierpińska (1987) has also discovered that 17-year-old humanities students found the equality $0.\overline{9} = 1$ perplexing. One of her students said the left-hand side would not equal one “no matter how many nines you put” and that it could not equal one unless “it goes to the very infinity” (Sierpińska, 1987, p. 379). Even some undergraduate analysis students who were aware of the definition of the repeating decimal “disregarded the definition” and argued that the left-hand side “was not equal to one” (Edwards & Ward, 2004, p. 417). Given these findings, our study investigates how pre-service teachers conceptualize adding two repeating decimals. In particular, we examine how adding two repeating decimals affects their understanding of the equality between repeating decimals and their rational counterparts. To frame this investigation, we turn to APOS theory because it offers one possible explanation for difficulties with repeating decimals, and we use it to analyze data from post-instructional assessments and describe the mental constructions involved in an individual’s conceptualization of the addition of two repeating decimals. Our study aims to answer the following research questions: *What do pre-service teachers attend to when adding two repeating decimals? How effectively does adding repeating decimals help pre-service teachers regard repeating decimals as objects rather than processes?*

Literature Review and Contextual Insights

Repeating decimals are geometric series in disguise, and some authors have investigated how students think about infinite series and their relationship to conceptions of infinity and limits. We situate our work within this literature, focusing on endless addition, series convergence, and other obstacles in understanding series.

On Endless Addition

Sierpińska (1987) found that students generally viewed infinity as an unending and uncompleted process while studying how students face epistemological obstacles involving limits. She explored erroneous uses of the geometric series and prompted students to look for errors in her reasoning. Likewise, Martínez-Planell et al. (2012) found that students view series as an endless addition process. They observed that students continued thinking of series as infinite summations after training. Their study shows that the rigorous definition of a series may only sometimes replace the conception of series as infinite additions in most students’ minds. Barahmand (2021) found that students resisted changing their intuitive use of equality and inequality signs despite facing series-related cognitive conflicts.

On Series Convergence

Alcock and Simpson (2004) suggest that some students go through a process similar to proofs and refutations because they conceive of a range of objects that satisfy the definition of convergence and debate which properties hold for all of them. They found that visual learners overestimated their understanding of the convergence of sequences and series, and they could not produce a written argument that a mathematician would consider appropriate and rate highly. However, Lindaman and Gay (2012) found that visualizing series convergence enables students to develop a longer-lasting mental image of convergence when they combine visualization with different calculus reform strategies. Fischbein et al. (1981) and Sierpińska (1987) found that some students developed implicit conceptions of convergence, but the students argued that the partial sums would never reach the limit but only get closer to it. Sierpińska (1987) also found that students could not conceive of infinite series equaling the limit of its partial sums, but some students began developing implicit conceptions of convergence. Likewise, Oehrtman et al. (2014) found that the guided reinvention approach allows students to construct a coherent definition of a sequence’s limit and define the pointwise convergence of an infinite series.

On Other Obstacles

Mamolo and Zazkis (2008) and Wijeratne and Zazkis (2016) found that students perceived infinity as an ongoing process even when they indirectly dealt with convergent geometric series. Martínez-Planell et al. (2012) also show that students conceive infinite series as an unending process of addition or as limits of partial sums, but some simultaneously hold both conceptions. Martin (2013) found that experts favour the formal definition of series convergence, while students rarely do. Alcock and Simpson (2005) found that some non-visual learners can manipulate symbols correctly but are prone to errors when they restate definitions, so they shallowly consider the meaning of their manipulations. However, Movshovitz-Hadar and Hadass (1991) and Sierpińska (1987) observed that students found that algebraicly manipulating geometric series could be perplexing and paradoxical. Although Barahmand (2021) found that students resisted changing their intuitive use of equality when dealing with infinite series, Oehrtman et al. (2014) found that guided reinvention helps students construct coherent definitions of pointwise convergence of series despite notational obstacles.

Theoretical Framework

Mathematics education researchers (see, for example, Alcock and Simpson (2004, 2005), Barahmand (2021), Fischbein et al. (1981), Lindaman and Gay (2012), Mamolo and Zazkis (2008), Martin (2013), Martínez-Planell et al. (2012), Movshovitz-Hadar and Hadass (1991), Oehrtman et al. (2014), Sierpińska (1987), and Wijeratne and Zazkis (2016)) have used various theoretical frameworks and methodologies to investigate students' conceptualization of the convergence of series and their limits, but APOS theory (Dubinsky et al., 2005a, 2005b), with the Totality stage, guided our investigation.

Dubinsky et al. (2013) explain that one possible way to describe a participant's ability "to see or to imagine all of the 9s [or other repeating digits] present at once" (p. 240) without seeing repeating decimals as objects is the existence of a new stage that they call totality, between the Process and Object stages. It refers to a student's ability to view all the repeating digits as present at once and simultaneously view this process as ongoing and completed. The fraction 1/3 illustrates this more explicitly because one divided by three is 1/3, whereas using long division yields the decimal representation of 1/3. Thus, this process is simultaneously ongoing and completed. In addition, participants' evidence of both progress toward and away from a stage suggested that there might be intermediate levels in the development from one stage to another. They note that their data indicates that seeing all of the nines present at once does not give this process the status of an object. According to the genetic decomposition related to repeating decimals Dubinsky et al. (2013) provide, repeating decimals become objects when participants can think of repeating decimals as numbers, perform arithmetic operations on them, substitute them for their rational counterparts, or conclude that they approach and reach their rational representations. Dubinsky et al. (2013) also revise the genetic decomposition related to repeating decimals by including three intermediate stages from the Process stage to the Totality stage: Start toward Totality (ST), Progress toward Totality (PT), and Emerging Totality (ET). They also propose three intermediate stages from the Process stage to the Object stage: Start toward Object (SO), Progress toward Object (PO), and Emerging Object (EO). Dubinsky et al. (2013) discuss the possibility of a mental mechanism existing "by which an individual moves from thinking of a process as a sequence of continuing steps to being able to imagine these steps all at once" (p. 251). APOS theory with the Totality stage postulates that the primary mental mechanisms for building action, process, object, and schema mental structures are interiorization for progress from action to process, de-temporalization from process to totality, and encapsulation for progress from totality to object (Dubinsky et al., 2013).

Method

Thirty pre-service secondary mathematics teachers from a mathematical methods course for teaching secondary mathematics participated in this study. At the time of data collection, our participants were in a professional development program in mathematics education. Their mathematics backgrounds and experiences varied significantly, but all held degrees in mathematics or science.

While exploring the topic of rational and irrational numbers, the participants used the definitions of rational and irrational numbers to relate them to terminating decimals, repeating decimals, and the decimal expansions of irrational numbers. The participants revisited various examples of repeating and non-repeating decimals and discussed converting a fraction to a decimal number. One student recalled the algorithm for converting a repeating decimal to a fraction by setting it equal to a variable, multiplying it by an appropriate power of ten, subtracting the former quantity from the latter, and solving for the variable. This suggestion led to a discussion about the meaning of digits and their positions, mainly focusing on repeating decimals. The students engaged in a lively debate, with differing opinions, while the instructors chose not to intervene to allow the discussion to unfold organically. Afterward, students worked in groups of two or three to discuss and respond to the following three questions:

Group Task: How would you add two repeating decimals? How would you check if your answer is correct? How would you respond to someone who says the sum of two repeating decimals is always a repeating decimal?

After that, we gave the participants a similar individual task.

Individual Task: A) How would you add two repeating decimals? How would you check if your answer is correct? How would you respond to someone who says the sum of two repeating decimals is always a repeating decimal? B) Now, add the following repeating decimals (without technology) before revising your initial responses. Please show all your work for each example: 0.252525... + 0.373737..., 0.434343... + 0.787878..., 0.858585... + 0.273273273..., 0.656565... + 0.587587587... C) After working on the above examples, please share your revised response to the three questions you answered earlier: How would you add two repeating decimals? How would you check if your answer is correct? How would you respond to someone who says the sum of two repeating decimals is always a repeating decimal?

The data for this study consisted of the participants' written group and individual responses. We categorized them by the participants' views on repeating decimals, focusing on their understanding of the underlying concepts. We analyzed their responses according to APOS's framework, focusing on intermediate stages from Process to Totality and Totality to Object.

Results

Some participants found handling infinitely many digits, converting repeating decimals to fractions, and doing column addition with repeating decimals challenging, and these particular challenges enabled us to categorize student responses.

Participants Face Challenges Handling Infinitely Many Digits

Many participants faced obstacles while checking if they had correctly added two repeating decimals in groups and their individual tasks. Many of our students viewed infinity as an unending and uncompleted process. Below, we discuss how this perception influenced their approaches. We hoped our students would see that decimal expansions contradict the notion of potential infinity because they are approximations that consecutively grow in time while representing a completed process. Still, only some of our students observed that. We will now explain how this perception influenced their approaches.

Initial responses. Many participants argued that when adding repeating decimals, using more repeating digits would allow them to get closer to the answer. They viewed the addition as an infinite process that continuously improves the final answer as the number of digits increases.

Starting points. In class, our participants could not immediately add repeating decimals as they would terminating decimals. Hence, they used truncated versions of repeating decimals or rational equivalents because they viewed both as numbers. Since repeating decimals did not have last digits, our participants initially resorted to operating on exact rational counterparts or rational approximations instead because they had last digits. Only one participant added two repeating decimals using the left-to-right addition by claiming that "3.777... + 4.666... = 7.000... + 1.300... + 0.130... + 0.013... + ... = 8 + 0.4 + 0.04 + 0.04 + ... = 8.444... = 8 4/9."

Participants Truncate Repeating Decimals to Verify the Conversion to Fractions

We found that many participants initially viewed the sum as an endless addition process, and they could not apply mental actions to the sum (or imagine applying them). Our data indicated that all the participants could easily convert repeating decimals into fractions using algebra. However, when we asked them how they would check if their final results were correct, most did not argue for the validity of algebraic conversion. Instead, they resorted to truncating repeating decimals, approximating the sum, and using calculators. Only a few students argued that the algebraic method was valid and that they only had to check if they added the corresponding fractions correctly. We now showcase some responses exemplifying the intermediate stages between Process and Totality.

Totality stage. Some participants built on their initial responses and displayed the PT conception as they were trying to imagine all the digits present at once by physically seeing many digits. One group first "added 3.99999 to 2.22222 and got 6.22221." Then they tried "adding 3.9999999999 to 2.2222222222 and got 6.2222222221." They concluded by saying, "There is a pattern here; the sum of our repeating decimals is also repeating; the more you extend them to add, the longer the decimal of the sum will be." Another group suggested adding two repeating decimals "using regular addition," but "the answer was inaccurate." They noted that adding more digits gave "a more accurate answer" and allowed them to "see what decimals would repeat." They concluded that the more digits we use, "the closer we get to the true answer." Another group suggested using a calculator and "increasing the number of repeating digits." They started with "0.333 + 0.666 = 0.999" and then suggested using "0.3333333333 + 0.6666666666 = 0.999999999" because they wanted "to show that the answer gets closer and closer to 1 (like a limit)." Another group said we could check our work by "doing the long addition with several decimal places for the best estimation." Another group wrote that "1/3 + 1/11 = 11/33 + 3/33 = 14/33 = 0.424242…" and added that "0.33333 + 0.09090 = 0.42423" confirms their answer. Another group said, "We can check the answer by adding the decimals to several decimal points." For example, "3.333333333… + 0.7142857… = 4.0476190… = 85/21." The group added that "this shows that our approximation using fractions works except when the fraction would produce a whole number" and that "this stresses that decimals are only approximations for fractions." Another group said, "Using the rounded version of the original numbers truncated to an appropriate number of decimal places, we could find an approximate solution to compare with our exact answer." Another group said that "using the rounded version of the original numbers" allows us to use "an approximate solution to compare with our exact answer." On their individual task, most participants abandoned the notion that adding two repeating decimals is a continuous process that improves the accuracy of the final result when more and more repeating digits are involved. We categorized some responses as the ST conception because these participants suggested truncating to estimate the sum without saying that using more digits improves the accuracy of the result. One of these participants suggested checking the addition "by manually typing out several decimal places." Another proposed to "truncate to an appropriate amount of decimals" and "add them up to check." Another suggested estimating "the approximate result of adding two decimals by manually adding the decimals up digit by digit to a certain amount." Another participant proposed "manually doing the long addition with several decimal places." Another suggested "ensuring that the sum makes sense based on an estimate."

Object conception. During the group exploration, our participants generally believed that repeating decimals were not complete numbers. They often asserted that they were approximations or approached rational numbers. We now showcase some responses exemplifying the intermediate stages between Totality and Object. Only two groups exemplified the Object conception by explicitly stating that repeating decimals are numbers. One group said that "recurring decimals are rational numbers" and "operations like adding apply and are valid when trying to add the repeating decimals together." Some individual responses exemplified the transition toward the Object conception. One participant demonstrated the EO conception by suggesting adding repeating decimals "as if they were not infinite, as the addition causes a repetitive pattern." This participant recognizes that one may add two repeating decimals, but only as if they were terminating decimals, so the participant has yet to reach the Object stage. Another proposed that "it is best to round each decimal to a certain large number of digits (12, for example) and try to add them to look for a recursion pattern." This response is similar to the previous one, but because rounding aims to identify the repetend in the final answer, this participant operates on repeating decimals as objects. Ten participants mentioned performing addition or "long addition," but we categorize these responses as the PO conception due to the lack of elaboration.

Participants Encounter Cognitive Obstacles Doing Column Addition

Our participants initially viewed the sum of repeating decimals as a process and the sum of their rational counterparts as objects. To them, the addition of repeating decimals is a dynamic process, while the sum of two corresponding fractions is a static object. Such a conceptualization involves viewing one side as an approximation to the other rather than strict equality. The standard right-to-left addition convention has prevented most of our participants from reaching the totality stage immediately. In contrast, some participants reached the Totality stage using pattern recognition but did not reach the Object conception. We now exemplify both cases.

School conventions (Totality stage). On the individual task, only one participant added using left-to-right addition, but everyone else relied on the standard right-to-left addition. This participant said, "Since repeating decimals have no last digit, we must start adding from the left to do column addition until we recognize the pattern in the new number." The same participant wrote that "3.777... + 4.666... = 7.000... + 1.300... + 0.130... + 0.013... + ... = 8 + 0.4 + 0.04 + 0.04 + ... = 8.444... = 8 4/9." This participant imagined and acted on all the digits to add from the left, reaching the Totality stage and immediately progressing toward the Object conception because the participant identified the sum as "the new number."

Pattern recognition (Object conception). Some exemplified the EO conception in their individual responses. One participant suggested "writing out the first several digits (enough to have the recurring part repeat at least three times) of each decimal and then adding them." The same participant said, "After seeing some repetition in my answer, I would assume that is the part that repeats and write the answer as a repeating decimal." This elaboration demonstrates the EO conception. The participant does not refer to the sum as a number, so we cannot conclude that this participant has reached the Object stage, but this conception is emerging. Another participant proposed to "ignore the final digit and determine the answer from the repeating portion since as the number of decimal places approaches infinity, the error tends to 0 and becomes negligible." The participant exemplifies the EO conception because the participant truncates to "determine the answer from the repeating portion" and recognizes the possibility of the sum being a number "as the number of decimal places approaches infinity." During the group exploration, only one group suggested checking "if adding the digits involves carrying ones." If it does not, "add some of the digits and look for a pattern." For example, "0.121212… + 0.656565… = 0.777… = 7/9." They added that if it does, add some digits and "look for a pattern but ignore the last digit." Since it is unclear whether this group recognizes the sum as a number or procedurally converts a repeating decimal to a fraction, we also classify this response as EO conception.

Conclusion and Discussion

Our paper aimed to investigate what pre-service teachers attend to when adding two repeating decimals and how effectively adding them helps them regard repeating decimals as objects rather than processes. Our participants demonstrated proficiency in converting repeating decimals to rational numbers and adding them that way. However, when the group and individual tasks asked them how they would verify their calculations, the majority predominantly truncated or approximated repeating decimals.

Unsurprisingly, the written group responses indicate that our participants initially viewed repeating decimals as processes that only approximated the corresponding rational numbers. The absence of the last digit and the reliance on the right-to-left addition posed the major obstacle.

Using the left-to-right addition, recognizing the sum's repetend, and stating that the final answer is a number rather than an approximation was a way of conceptualizing repeating decimals as numbers. More research could uncover the role of left-to-right addition. Most surprisingly, our data indicates that almost all the students have reached the Totality stage. A few reached the Object stage, and many came close to viewing repeating decimals and their sums as objects. Asking participants to think about how they would check that they had added two repeating decimals correctly inadvertently helped them reach the Totality stage and even the Object stage, and herein lies our main contribution to the research on repeating decimals. The data suggest that the unusual task we designed helped most participants reach the totality stage. Moreover, our study indicates the existence of the mental mechanism of de-temporalization for transitioning from process to totality, as almost all of our participants have reached it. Prompting participants for ways to check if they had correctly added repeating decimals led them to think about how to perform column addition with repeating decimals. Our participants seem to have reached the Totality stage because they eventually realized that they had to see and act on all the digits as they would when doing column addition with terminating decimals. Future research would help the mathematics education community ascertain if this is the determining factor.

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