The Role of Exploring and Conjecturing When Introducing Derivatives in Calculus: To Prove or to Verify?

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In this report, I present how eight experienced college calculus instructors used exploring and conjecturing to frame their instructional tasks for introducing derivatives. During semi-structured interviews with each instructor, I prompted them to propose instructional tasks for introducing derivatives. The tasks were broken down into their smallest calculus-specific problems, which I refer to as calculus-specific task units or CTUs. Using the notion of instructional situations by Herbst (2006), I categorized the exploring/conjecturing CTUs into a single instructional situation: Exploring or conjecturing to verify statements or relationships. I further describe how the CTUs were invoked in the tasks and the kinds of mathematical works they were expected to engage students in. To conclude, I discuss why the exploring/conjecturing CTUs revealed in the study either did not, or could not, converge to proving situations.

Keywords: Calculus, exploring and conjecturing, derivatives, instructional situations, tasks

Mathematical disciplinary practices, or activities commonly undertaken by professional mathematicians—such as exploring, conjecturing, defining, and proving (Rasmussen et al., 2005)—have been promoted to enhance students' engagement with mathematics in the classroom and improve their conceptual understanding. In the context of calculus education, the reform movement of the 1980s led to emphasizing conceptual learning and active student engagement in the classroom by relying on graphical, numerical, and technological elements for exploring and illustrating ideas, while reserving algebraic manipulations more for computational purposes (Bressoud et al., 2016; Kidron, 2020). Researchers, mathematics departments, and individual calculus instructors continue to explore, study, and implement these innovations to improve student outcomes in calculus (Borji et al., 2018; Bressoud et al., 2016; Code et al., 2014; Gerami et al., 2023).

In this paper, I focus on how college calculus instructors utilize the disciplinary practices of exploring and conjecturing. I define exploring as examining mathematical objects and relationships among them, and conjecturing as a particular case of exploring, during which one produces a generic or conceptual statement after exploring relationships among some mathematical objects. To better capture the learning opportunities the instructors create in their classrooms, I turn my attention to one concept—the derivative—examining how instructors present it through their instructional tasks (i.e., problems and activities designed to "focus students' attention on a particular mathematical idea;" Stein et al., 1996, p. 460). This paper addresses the following research question: What roles do exploring and conjecturing play in calculus instructors' instructional tasks when introducing the concept of derivatives?

Theoretical Framework

By conceptualizing teaching as a social activity, Brousseau's (1997) *theory of didactical situations in mathematics* assumes that every mathematics class operates under a didactical contract, referring to "shared expectations of what it means to study mathematics in school" (Herbst et al., 2023, p. 402). Although often implicit, the contract defines the roles of the teacher and students, governs the classroom relationships, and regulates the work of teaching and learning mathematics through social content-related norms (Brousseau, 1984).

Relying on the notion of didactical contract, Herbst (2006) introduced *instructional situations* as local or specialized didactical contracts for specific instructional exchanges within a course of study. Herbst et al. (2020) defined *task framing* from the perspective of the teacher in mathematics instruction as choosing a way of handling a problem about the knowledge at stake; once recognized by students, the task framing allows students to know what kind of mathematical work they should perform and what kinds of interactions they can have. In other words, for instruction, teachers choose how to frame a problem, so that students know the task at hand and how they are expected to handle the problem. For example, Herbst et al. (2023) provided an example of how a teacher, with respect to the same piece of knowledge—the tangent segments theorem that states "a circle is tangent to two intersecting lines if and only if the points of tangency are equidistant from the point of intersection" (Herbst et al., 2023, p. 403)—can both frame the work as a construction and as a proof. In this way, instructional situations are both each of the distinct problem framings used in a course of studies, as well as "the specific norms that regulate the teacher's and students' work on them" (Herbst et al., 2020; 2023, p. 402). Researchers have identified various customary instructional situations, or distinct types of framing tasks, mainly in high school geometry and algebra (e.g., exploring or examining a figure, shape, or solid constructing; constructing a shape, figure, diagram, or solid including physical models; geometric algebraic calculation; solving equations in one variable). This line of work highlights that teachers can select from the available set of instructional situations to frame their instructional tasks, consequently launching, organizing, and scaffolding students' work (Herbst et al., 2018; 2020).

Methods

The data for this report comes from a larger study (Gerami, 2024) in which I purposefully selected and interviewed eight experienced calculus instructors from across the United States, who reported using various patterns of inquiry in their teaching. The instructors chose their own pseudonyms: Justin, Adrian, Barry, Matthew, Gopher (he/him); Monica (she/her); Max (they/them); and Alex (any pronouns). During four semi-structured 1–2-hour long interviews with each instructor, the instructors were asked to propose eight tasks for introducing derivatives. To ground the instructors' proposed tasks in students' conceptions of derivatives, I organized the interviews using Zandieh's framework (2000, Table 1) in two ways: 1) each interview was dedicated to one representation of the derivative (graphical, verbal, physical, or symbolic), and 2) The instructors were asked to propose two tasks within each representation that would help transition students' conception from one process-object layer to the next (the first task from ratio to limit, and the second task from limit to function).

Table 1. Zandieh's (2000) adapted framework for the concepts of derivative.

		Representations			
		Graphical	Verbal	Physical	Symbolic
Process-Object Layer	Ratio	Slope of the secant line	Average rate of change	Average velocity	Difference quotient
	Limit	Slope of the tangent line	Instantaneous rate of change	Instantaneous velocity	Limit of the difference quotient
	Function	Graph of the derivative function	Rate of change of a function	Velocity as a function of time	Derivative as a function

The instructors collectively proposed 56 tasks. Because most of the instructional tasks comprised of multiple parts, I suspected that the instructors often used multiple calculus-specific instructional situations to frame the multi-part instructional tasks. Therefore, I broke down the instructional tasks into their smallest calculus-specific problems, which I call *calculus-specific task units* or *CTUs*. By this, I mean that if I further divided a CTU into smaller tasks, the results would not be recognizable as calculus tasks; instead, they would be recognized as tasks from algebra, arithmetic, or other content areas preceding calculus in a standard mathematics curriculum sequence. After finalizing the CTUs of all kinds (calculating, graphing, exploring/conjecturing, etc.), I used thematic analysis (Saldaña, 2021) to achieve two main objectives: 1) identify the calculus-specific instructional situations, as defined by Herbst and Chazan (2012), by reducing the list of CTUs; and 2) synthesize the findings. During these processes, I heavily relied on Herbst et al.'s (2010) descriptions of instructional situations developed in geometry to guide myself on what to pay attention to within each instructional situation. To identify the calculus-specific instructional situations, I asked myself *how can I identify meaningful new categories for them if I looked past the specific mathematical objects within each CTU? How are the mathematical work and the norms of such work that students are expected to do similar/different if I looked past the specific mathematical objects within each CTU?* Essentially, to define the instructional situations, I differentiated between and categorized different CTUs based on the nature of mathematical work and the norms that regulated such work by relying on my own knowledge as a calculus instructor. This process resulted in five calculating, one exploring/conjecturing, two graphing, two installing, one proving, and one solving equation instructional situations (see Gerami, 2024 for complete description). I foregrounded each instructional situation across all instructional tasks to uncover patterns and themes about situation use. In doing so, I gathered tasks with similar codes for comparison to address the following questions: *How did instructors invoke these situations? How frequently and diversely were they employed? In what sequences and patterns did situations from this category typically appear in the tasks?* In this study, I focus only on the exploring/conjecturing findings.

Findings

All instructors used exploring/conjecturing calculus-specific task units (CTUs) in their instructional tasks for introducing derivatives. Given that conjecturing is a particular case of exploring, indicating when students are expected to produce hypotheses based on their perception (or contextualized conjectures) or reasoning (or reasoned conjectures), I included them in one category. I identified 11 distinct exploring/conjecturing CTUs across all the proposed instructional tasks. These CTUs with their frequency of use are listed in Table 2 (see Gerami, 2024 for detail of how the instructors used them in their tasks). I reduced all the exploring/conjecturing CTUs to one instructional situation: *Exploring or conjecturing to verify statements or relationships*. This instructional situation requires students to confirm that a statement or relationship is true, either based on perception or by making connections between multiple mathematical representations, without deductively proving it.

Table 2. Exploring/Conjecturing CTUs identified across the tasks, their engaged layer of derivative, and their frequency of use

Layer	Exploring or conjecturing CTUs to verify statements or relationships	*n*
Ratio	**E1.** Exploring the pattern among rates of change on consecutive intervals of one unit of length	1
	E2. Conjecturing the relationship between average rate of change of a function on an interval and the slope of the secant line	1
Limit	**E3.** Conjecturing the existence and/or value of instantaneous velocity or slope of a tangent line or a best estimate for rate of change at x_0 • given a table of values for various x_i and $f(x_i)$ including x_0 and values to the right and left of x_0 • between x_1 and x_2 (to the left and right side of x_0), given a table of values for various x_i and $f(x_i)$ including x_1 and x_2 • from a pattern of slopes of secants h-units distanced to the right side of x_0, with h decreasing • from a pattern of slopes of secants to the left and right side of x_0 • from a pattern of Δx-units distanced to the right side of x_0, with Δx decreasing with formal limit definition $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ • using average rates of change from a pattern of Δx-units distanced to the right side of x_0, with Δx decreasing	8
	E4. Conjecturing a formula for instantaneous velocity at x_0 from a pattern of Δx-units distanced to the right side of x_0, with Δx decreasing with informal limit definition $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ as $\Delta x \to 0$ with formal limit definition $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$	3
	E5. Exploring the slope of a tangent line of a cusp	1
Function	**E6.** Conjecturing how a smooth function's graph looks like if we zoom in enough	2
	E7. Exploring the graph of difference quotient $\frac{f(x+h)-f(x)}{h}$ with sliding h going to zero	1
	E8.1. Exploring the relationship between graphs (of $f(x)$ and $f'(x)$) **E8.2.** Conjecturing the algebraic relationship between graphs of $f(x)$ and $f'(x)$ for $f = x^n$ (e.g., f is quadratic implies that f' is linear)	4
	E9.1. Exploring the physical journey of an object given the graphs of $f(x)$ and $v(x)$ **E9.2.** Exploring the physical journey of an object given the graph of $v(x)$	2
	E10. Conjecturing the shape (trend—i.e., increasing, decreasing, constant—and concavity) of a function given the sign (positive, negative, zero) and change (increasing, decreasing, constant) of its instantaneous rate of change	1
	E11.1. Exploring $f'(x)$ using patterns of $f'(x_0), f'(x_1), f'(x_2), \ldots$ **E11.2.** Conjecturing the power rule ($f'(x)$ for $f(x) = x^n$ for all integers) using the patterns of derivatives of $f = x$, $f = x^2$, $f = x^3, \ldots$	4

Note. The different terms that instructors used in their tasks to refer to the same concept are wave underlined to indicate their equivalence.

At times, the instructors used explicit expressions, such as *explore*, *hypothesize,* and *make a prediction*, in their task descriptions to evoke these CTUs (e.g., "Can you make a prediction about what will happen when you zoom in on any point on this graph?" Justin, Prompt 5's task). However, they more often directed students towards the object of inquiry by asking them: whether they could do something (to imply the possibility of performing a mathematical action, rather than assessing students' mathematical abilities), to explain their thinking, or to consider what might occur under specific conditions (e.g., "Is there a 'best' AROC [average rate of change] estimate?"; Alex, Prompt 5's task; "Can you write down the exact number of the speed at $t = 2$?", Matthew, Prompt 1's task; "Talk about the relationship between these average velocities and what [do you] think would happen if [the windowpanes became smaller?]", Monica, Prompt 1's task). The rather indirect use of language to evoke exploring/conjecturing situations suggests that instructors may have to clarify the situation in class to convey to students what they are expected to do.

The instructional situation was presented in the form of two learning opportunities in the instructional tasks:

a) Exploring graphical representations of derivatives to make sense of, or conjectures about, symbolic representations and relationships between, functions, derivative values, and derivative functions (E2, E5, E6, E7, E8.1, E8.2, E9.1, E9.2, E10), and
b) Inductively constructing symbolic representations (including definitions) to model derivative patterns (E1, E3, E4, E11.1, E11.2).

The first group—exploring graphical representations of derivatives to make sense of, or conjectures about, symbolic representations and relationships between, functions, derivative values, and derivative functions—includes CTUs in which students are expected to make connections between graphical and/or symbolic representations of objects that are related to derivatives. In these CTUs, students rely on graphs of functions, secant and tangent lines, and derivative functions to explore and conjecture how they are related to one another or to make sense of numerical or symbolic representations related to derivatives, such as the derivative value at a cusp at in E5 and the limit of the difference quotient in E7.

The second group—inductively constructing symbolic representations (including definitions) to model derivative patterns—includes CTUs in which students are expected to detect mathematical patterns about changes and/or derivatives among some mathematical objects, and to possibly come up with procedures or formulas that model the pattern. These situations, like E3, E4, and E11, can precede installing (i.e., defining) situations by allowing students to develop intuition about definitions or formulas before they are formally installed.

I identified two recurring patterns in how exploring/conjecturing CTUs were integrated into the instructional tasks: after requiring students to calculate and graph things, or before installing a new definition, formula, or procedure. In the first pattern, the instructional tasks with exploring/conjecturing CTUs required students to perform calculations and graph things via calculating and graphing CTUs. Subsequently, students were prompted to explore or make conjectures. For example, in Alex's task for Prompt 1, students were initially asked to use average velocities to estimate instantaneous velocity (C1 and C4). They were then asked to conjecture a formula for instantaneous velocity at a point by "continuing to make their estimate better." Similarly, in their tasks for Prompt 3, Max, Alex, Adrian, and Monica used combinations of graphing and calculating CTUs related to average or instantaneous rate of change at a point before E3, in which they asked students to conjecture about the existence or the value of instantaneous rate of change at a point. In another task for Prompt 5, Gopher first asked students to estimate the rate of change at a point, and then followed the instructional task by asking students to use an applet to conjecture the slope of tangent line at a cusp (E5). As these examples demonstrate, it appears that these preceding calculating and graphing CTUs would make room for students to observe, visually inspect graphs or sequences of numbers and identify patterns. Thus, in these exploring/conjecturing CTUs, at stake are making sense of the graphs and/or calculations, connecting various concepts/representations of derivatives, or discovering how graphs or calculations would change under specific conditions. Although the role of the teacher in exploring/conjecturing situations is often to give students the object of explorations and possibly legitimize the resources and tools that students can use (Herbst et al., 2010), in these cases the work of the teacher was more directive because the tools and resources were already given in the instructional task via students working on them through explicit graphing/calculating CTUs beforehand. This limits students' free choice in selecting the representations and operations for their exploration. There were only a few instances where the exploring/conjecturing CTU did not proceed any other CTUs. Although this suggests that students were granted more autonomy, the instructional tasks already provided the related representations for their explorations (e.g., Alex's task for Prompt 6; Justin's task for Prompt 4).

In the second pattern, some exploring/conjecturing CTUs led to CTUs that stated a new definition, formula, or procedure or validated students' findings from the exploring/conjecturing CTUs for future use. For example, in their instructional tasks proposed for Prompt 5, after asking students to conjecture the existence of instantaneous velocity at t_0 given a table of values for various t_i and $f(t_i)$ with Δt decreasing (E3), Matthew used E4 to ask students to conjecture a formula for instantaneous velocity at t_0 using their explorations from E3, and Alex used I4 to formally install instantaneous rate of change. In another example, Alex and Monica installed the power rule (I7) in their instructional tasks for Prompt 8, after having students calculate derivatives of $f = x$, $f = x^2$, $f = x^3$, … (C7 and C11) and conjecture a formula for the derivative of x^n (E11). Thus, in this pattern, the exploring/conjecturing CTUs seem to create opportunities to legitimize the new knowledge gained from the CTUs. Therefore, at stake in these CTUs is introduction to or becoming familiar with new general formulas and procedures about derivatives (limit definition of instantaneous rate of change at a point, finding slope by zooming in, power rule).

Discussion

In this report, I demonstrated that although the calculus instructors in the study relied on a diverse and engaging range of exploring/conjecturing calculus-specific task units (CTUs) in their instructional tasks when introducing derivatives with inquiry, the CTUs were all to verify statements or relationships. Despite the instructors' reliance on these CTUs when framing their tasks, the instructional situation either did not, or could not, converge to proving situations. Although Herbst et al. (2010) asserted that exploring/conjecturing situations in geometry "often converges to the formulation of conjectures, or the proofs of those conjectures," it seems that calculus instructors in this study use exploring/conjecturing CTUs to have students reason abductively and inductively. It seems that in the CTUs identified in this study, deductive inferences (i.e., proving) were either: not appropriate given students' existing knowledge (e.g., proving E6 without students' knowledge of infinitesimals), not called for because these CTUs were designed to lead to installing situations (which could not be proved deductively; e.g., E3 and E7), or that even when proving conjectures were possible, the instructors chose to not pursue them. Regarding the latter category, I suspect that six CTUs could lead to proving situations that student could work on with using their existing mathematical knowledge:

- E2: Conjecturing the relationship between average rate of change of a function on an interval and the slope of the secant line—could lead to proving that the two quantities are equal.
- E5: Exploring the slope of a tangent line of a cusp: proving that the derivative at the cusp does not exist—could lead to demonstrating that the right-hand limit is not equal to the left-hand limit.
- E8.2: Conjecturing the algebraic relationship between graphs of $f(x)$ and $f'(x)$ for $f = x^n$)—could lead to proving that $f'(x) = nx^{n-1}$ using a limit definition of the derivative, indicating that the function loses a power when differentiated.
- E10: Conjecturing the shape (trend—i.e., increasing, decreasing, constant—and concavity) of a function given the sign (positive, negative, zero) and change (increasing, decreasing, constant) of its instantaneous rate of change—could lead to proving what happens to slope of the tangent line, written using the limit definition, at a random point on any of those intervals.
- E11.1: Exploring $f'(x)$ using patterns of $f'(x_0), f'(x_1), f'(x_2), \ldots$—could be followed by using a limit definition of derivative to calculate $f'(x)$.
- E11.2: Conjecturing the power rule ($f'(x)$ for $f(x) = x^n$ for all integers) using the patterns of derivatives of $f = x$, $f = x^2$, $f = x^3$, …—could lead to proving that $f'(x) = nx^{n-1}$ using a limit definition of the derivative.

Of these CTUs, however, only three—E8.2, E10, and E11.2—were about general statements. While the other CTUs could also lead to proving, they were specific to particular functions and the conclusions would not always be useful for other functions. The absence of deductive reasoning following the CTUs can be attributed to the CTUs being either about particular examples, or because when they were generic enough, the instructors found inductive reasoning more appropriate than deductive reasoning, possibly due to time-consuming or challenging algebraic manipulations.

Indeed, the instructors in this study used a high number of exploring/conjecturing CTUs, which aligns with associating inquiry in undergraduate mathematics education with authentic problem solving and the exploration of novel ideas. However, despite the emphasis on conceptual mathematical reasoning in inquiry-oriented contexts, instances where exploring/conjecturing situations could have been followed by reasonable proving situations were, in fact, not followed up with proving situations. One explanation could be that instructors perceived proving situations, especially with student-centered activity structures, as too challenging for calculus students. Paradoxically, it may be the case that more lecture-oriented instructors might use more proving situations via lecturing. It is also possible that college instructors simply do not consider calculus an appropriate subject to engage students in the work of proving, given that 'formal' proving is often introduced in linear algebra, following the calculus series. Nonetheless, my coding of proving CTUs was rather broad, encompassing deductive mathematical justifications, which researchers have shown could be done in any mathematics class (Stylianides, 2007).

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