Reasoning Beyond Content-Specific Objects: A Case Study of One Student's Structural Thinking

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In all levels of mathematics, especially beginning at the undergraduate level, structural thinking is pivotal for deeper understanding or more complex reasoning about the relevant mathematical concepts. In this preliminary research study, we investigated the structural thinking of one advanced undergraduate mathematics student as he conjectured the definitions of various concepts in graph theory. We observed various affordances and obstacles to his structural thinking. Affordances included his attention to a general structural property and his use of analogical reasoning with corresponding structures in group theory and point-set topology. Obstacles included utilizing a constrained example space and relying on his familiarity with previously encountered mathematical concepts.

Keywords: Abstraction, Analogy, Graph Theory, Structural Thinking

The ability to think *structurally* in undergraduate mathematics is an invaluable skill. Spurred by David Hilbert’s insistence on conforming to a structural approach to studying mathematics in the early 1900s, the evolution of mathematical thinking to conform to the flexible recognition of underlying structure of objects has been a purposeful endeavor in modern research (see Hausberger, 2018). The shift toward structural thinking also played a significant role in the development of the mathematical subjects we offer to undergraduate students today. Take, for example, the subjects of abstract algebra and point-set topology. Before the expansion of structural thinking in the early 1900s, such subjects were difficult to access for both students and professional mathematicians alike. Despite the modern focus on structure in mathematics, educational research has revealed that students struggle to articulate certain concepts often associated with structural thinking, such as quotient group (Dubinsky et al., 1994). However, we wondered about the development of students’ *structural understanding* as a separate phenomenon as opposed to their content-specific understanding. In this paper, we present a preliminary investigation of what one student, Isaiah (a pseudonym), understood about structure after having taken courses in introductory abstract algebra and point-set topology.

Literature Review and Conceptual Framework

At the center of this research study are the notions of structural thinking, analogy, abstraction, and the deep connections between them. Mac Lane (1996) stated, “Structure is essentially a list of mathematical operations and relations and their required properties…, and often so formulated to be properties *shared by a number of possibly quite different specific mathematical objects*” (p. 174, emphasis added). In this rough definition, there is already implicit reference to a type of relational structure between mathematical objects that share certain properties.

To investigate structure more directly, Linchevski and Livneh (1999) introduced the notion of structure sense, which aimed to capture one’s ability to recognize the same structure flexibly and creatively across various contexts, specifically in the context of high school algebra. Novotná and Hoch (2008) extended this concept to structure sense in abstract algebra, further elaborating on the structure inherent to the set and binary operation of algebraic structures such as groups. Recently, Hicks and Flanagan (2024a) further extended structure sense to more targeted structures, including sub-structure sense, sense of structure preservation, and quotient structure sense (see also, Hicks & Flanagan, 2024b). Additionally, to provide a way to analyze students' reasoning across related structures (e.g., subgroup and subring), we introduced analogical structure sense, which aimed to "parse students' attention to structure during analogical reasoning" (Hicks & Flanagan, 2024a, p. 4).

Hicks, Flanagan, and Park (2023) investigated the relationship between analogy and abstraction, which culminated in two ways that students may abstract structure during the process of analogical reasoning (see also, English & Sharry, 1996). This suggested that a certain level of abstraction may be required when engaging with analogical structure in widely different domains, such as between groups and topological spaces. This preliminary study aims to further contribute to this growing body of literature on the connections between structural thinking, analogy, and abstraction, by investigating how one advanced undergraduate student reasoned about developing structures in graph theory. To do this, we proposed the following two research questions:

1. What were the affordances and obstacles that an undergraduate student elicited while conjecturing the definitions of structures (sub-structure, structure preservation, quotient structure) in a new context of graph theory?
2. In what ways did the student utilize his prior experience with analogous structures in group theory and topology when conjecturing the definitions of structures in graph theory?

Methods

Because our goal was to investigate students' structural thinking after having taken courses in higher mathematics, we solicited participation from all students in an introductory point-set topology course in which abstract algebra was a prerequisite. Of the eight students that were contacted, only Isaiah responded. The data for this study was collected from a one hour, task-based interview with Isaiah, conducted by both authors digitally over Zoom. The interview was intended to serve as a pilot for future data collection and comprised of four distinct parts: a brief introduction to the concept of a graph and a determination of what Isaiah already knew about graphs from his previous coursework, followed by requesting Isaiah to create each of the concepts of subgraph, graph homomorphism, and quotient graph. During the interview, we asked clarifying and probing questions when needed.

The interview was subsequently transcribed and analyzed by both authors. While Isaiah was creating his graph-theoretic structures, we found that he presented several affordances and obstacles to his structural thinking. Affordances for Isaiah's structural thinking included features of his thinking that helped drive him forward in creating structure or reasoning about underlying structure. By contrast, obstacles included features of Isaiah's thinking that led to impeded creation of structure or resulted in incomplete or inconclusive reasoning about underlying structure. We note here that we did not directly attend to Isaiah's graph-theoretic thinking, meaning that we were not looking for affordances and obstacles to his understanding of concepts in graph theory. Rather, we wished to capture the affordances and obstacles to his more general thinking about the underlying structure of the graph-theoretic concepts. In this way, it was possible to identify affordances for Isaiah's structural thinking that would perhaps be considered obstacles to his graph-theoretic thinking. Likewise, certain aspects that might be affordances to his graph-theoretic thinking (i.e., recalling what he knew about graphs from his previous coursework), might have also resulted in obstacles to his structural thinking.

Results

The analysis revealed two primary affordances and two primary obstacles to his structural thinking about the three graph theoretic concepts. We provide individual evidence for each of these affordances and obstacles. As stated above, our analysis attended to structural thinking rather than his content-specific thinking about each of the relevant graph theoretic concepts.

Attention to a General Structural Property as an Affordance

Throughout the interview, Isaiah attended to a general, overarching property that described the structure of graphs. For example, when developing the concept of a subgraph, Isaiah stated, "So, I think there would have to be... vertices still being adjacent in the subgraph." This was the first indication that Isaiah identified adjacency as the overarching structural property for graphs. The act of establishing this property proved to be an affordance to Isaiah's structural thinking throughout the interview, specifically by helping him to make headway with establishing access for creating the subgraph, graph homomorphism, and quotient graph structures. For example, when working on the graph-theoretic homomorphism concept, Isaiah stated:

> So, when I hear of homomorphism or homeomorphism, it's the idea of structure preserving, or you know, isomorphism. So, in a graph, I guess this structure is what stuff is adjacent to, if that makes sense, I think. So, I'd have to think about how to define it so it actually makes sense.

Furthermore, present in Isaiah's structural thinking was also a fair amount of flexibility (see Lee & Heid, 2018) for what constituted the general structural property for graphs. Specifically, he was open to revising his understanding of the general structural property as he continued his constructions of graph-theoretic concepts. While he initially focused on adjacency as the primary property, he slightly modified this property as he explored the graph homomorphism concept:

> I'm trying to think, because I probably want to just look at stuff, what it's adjacent to, and then have to work through the graph in order to define the homomorphism. So, I would have to send one vertex to a vertex in another one [graph] such that it has the same number of neighbors... Yeah, I guess that's, that makes sense. I think, if I say, a graph is homomorphic if there a function, say f , going from the vertices of one to the vertices of another one such that, and then I'll say, if, I've seen this notation before, so if the [number of] neighbors of V is equal to the [number of] neighborhoods of f of V , so $N\ V$ $[N(V)]$.

Thus, Isaiah's structural thinking was afforded by first establishing a central structural property for defining graphs which subsequently undergirded his graph-theoretic constructions. We note here that although Isaiah was technically shifting away from the normative graph-theoretic property of graph homomorphism itself, his structural thinking was afforded by his attention to the general structural property, as well as his openness to revising the criteria as he progressed.

Analogical Reasoning as an Affordance

During his reasoning about graph-theoretic concepts, Isaiah's structural thinking was afforded through analogical reasoning with group theory and topology. While constructing the subgraph concept, Isaiah noted the following:

> But I guess because I mean in both of them, it's all the elements are coming from whatever, a group, a subgroup, all the elements are also elements of the group. In the

> topological space, at least the set, all the elements are coming from the set... I think this [subgraph] is more similar to a subgroup type of definition where that makes sense.

Here, Isaiah is recognizing that in the same way that the subgroup and subspace concepts are defined by a subset relation, the subgraph concept also possesses the subset relation. In this way, Isaiah had constructed the subgraph concept by first identifying that there should be some notion of a subset involved. However, Isaiah also attended to specific differences, further emphasizing the role of analogical reasoning in his structural thinking. In particular, he made an observation regarding how the subgraph was related to the parent graph structure. He stated the following:

> So, in a subgroup, the binary relation, you're still going to have that I guess. And then a subgraph, you don't need to include all the edges, I think, so it's a little bit different because you can just remove one edge from the graph and that still should be a subgraph. But, in a group, you can't remove a binary [relation], you can't remove one of the relations, I guess, and still have it be; it wouldn't be a group then. So, I guess there's a little bit more freedom in a subgraph than in a subgroup or a subspace.

Isaiah appeared to make an analogical connection between the edges between vertices in a graph and the binary operations between elements in a group. By attending to this, he observed that there was "more freedom" when determining subgraphs as opposed to determining subgroups. The fact that Isaiah not only leveraged similarities between contexts, but also made fruitful observations about the differences between contexts, meant that Isaiah's analogical reasoning served as an affordance to his structural thinking.

Constrained Example Space as an Obstacle

Throughout the interview, Isaiah appeared to rely on a constrained collection of examples for graphs. In particular, whenever there was more than one graph present in his reasoning, he almost always chose the second graph to be a subgraph of the first graph. This proved to be an obstacle for his reasoning about the structure of graphs, and specifically the meaning of structure preservation. While testing his definition of graph homomorphism, which he conjectured to require preservation of the number of neighbors, he stated:

> Let me try and see if this definition even holds up with what I'm thinking exactly...Maybe I'll try and do a subgraph of this. I think that may be a better approach...Actually, I think I should change this definition a little bit to be bijective too.

Since he only looked at examples with subgraphs, his current working definition of graph homomorphism would always be bijective. In particular, by attending to only the specific examples including subgraphs, Isaiah was no longer thinking about the underlying structure of graphs, but rather only considered a special case where bijectivity was always present. Even though this was correct reasoning from Isaiah in the context of this example, he did not consider how this addition of structure impacted his definition beyond those cases where subgraphs were present. Instead, he appeared to forcibly add an extra condition due to the result of a single constrained example. Thus, this presented an obstacle to Isaiah's structural reasoning because rather than flexibly considering the validity of his original definition, he chose to add conditions to his definition to satisfy his limited set of examples.

Prior Mathematical Familiarity as an Obstacle

Multiple times during the interview, Isaiah would reference mathematical concepts he had encountered before. Although prior familiarity with concepts can act as an affordance for structural thinking, such as was the case with Isaiah's analogical reasoning about previously known concepts in the domain of group theory and topology, we found that an obstacle also arose from his desire to inject familiar concepts into unfamiliar situations. A primary example of this included when Isaiah was working through his definition of graph homomorphism. He recognized the importance of preserving the structure of adjacency but kept returning to the number of neighbors because he had “seen this notation before.” For instance, he stated:

> Say you have two edges that are both adjacent in the one graph and you apply the function, I think they should end up being adjacent in the other graph as well. I think I could probably prove that by. Yeah, no. So, I don’t know. I think the structure and graph is probably pretty, very related to I guess the number of neighbors that stuff has.

Similarly, when working through the concept of quotient graphs, Isaiah expressed how he had seen the “power set of graphs” before, which primarily guided his construction. Unlike Isaiah’s reasoning by analogy, which involved calling forth a more tightly connected collection of structural knowledge about group theoretic and topological structures, we found that Isaiah’s reasoning in the above examples relied on singular concepts disconnected from any structural foundation. Thus, Isaiah’s structural thinking was limited by the concepts he used to conjecture the definitions that were called from memory with no appeal to their structural purpose, causing his prior mathematical familiarity to act as an obstacle to this kind of thinking.

Discussion

The results and analysis from this preliminary study shed further light on the importance and value of structural thinking, especially at the undergraduate level. Beyond this, the results provide evidence that content-specific reasoning and structural reasoning do not always coincide. Particularly, an affordance to one’s structural thinking may be an obstacle to their content-specific thinking, and conversely. Isaiah used meaningful, constructive structural reasoning without the correct graph-theoretic definitions. We conjecture that an affordance to both one’s structural and content-specific thinking would help generate effective, meaningful learning. However, a more detailed study would be required to address this. Moreover, this study further supported the value of students leveraging analogical reasoning when engaging with new-to-them mathematical concepts (Hicks & Flanagan, 2024a; Lee & Sriraman, 2011). Also, if students can meaningfully construct an abstracted structure for certain concepts, such as sub-structure, they can use that structure to reason structurally in a new context (Hicks et al., 2023).

The results of this study also provided support that learning and understanding concepts in abstract algebra and point-set topology are difficult for undergraduate students (e.g., Larsen, 2009; Melhuish et al., 2020; Serbin, 2023), especially quotient concepts, like quotient groups (e.g., Hicks & Flanagan, 2024b). However, in contrast, these results also revealed that students can generate productive structural reasoning, despite having challenges with the content-specific definitions. In addition, the results highlighted the importance of a flexible example space for structural reasoning, especially in the context of conjecturing new definitions (e.g., Lynch & Lockwood, 2019). When reasoning through concepts with example-related activity, it is imperative that there exists coherence between an individual’s structural reasoning within the examples and the general structural property for the concept itself. Finally, Hazzan (1999) claimed that undergraduate students tend to reason with “familiar objects,” which can sometimes be an unproductive strategy for them (pp. 76-77). The finding from this study that prior mathematical familiarity can be an obstacle to structural thinking supports this claim.

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