Floating in a Most Peculiar Way: The Role of Feedback in a Playful Calculus Task

Robert Ely
University of Idaho

Amy Ellis
University of Georgia

Dru Horne
University of Georgia

Sohei Yasuda
University of Georgia

Anna Bloodworth
University of Georgia

We created and piloted a playful calculus-based activity called Major Tom with preservice secondary mathematics teachers over two classroom sessions. Players create sequences of jetpack boosts (accelerations), controlling an astronaut's movements to arrive at a pick-up location at a particular time. The learning goal was for students to draw upon and enhance their knowledge of integration to determine velocities and displacements. We found that the imprecision of the system's feedback inhibited the students' productive use of their formal calculus knowledge, while on the other hand an instructor-led structured discussion about velocity graphs supported it. Neither of these features of the playful math environment much affected the students' playfulness, but a third feature, the opportunity to create and solve your own challenges, fostered significantly increased play and laughter. This contributes to our broader goal of how to elicit play with undergraduates while simultaneously engaging and building their formal mathematics knowledge.

Keywords: velocity, integral, calculus, play, acceleration

Mathematical play is beneficial for students. It can improve enjoyment and engagement (Plass et al., 2013), increase equitable access to mathematics (Widman et al., 2019), and help students develop creativity and a disposition for innovation (Barab et al., 2010; Gresalfi et al., 2018). It has also been shown to support mathematical learning and reasoning at the elementary and middle-school levels (e.g., Edo & Deulofeu, 2006; Jasien & Horn, 2018; Levine et al., 2012). However, there are still significant gaps in our knowledge of how play can support learning, particularly at the undergraduate level. The aim of our work is to support undergraduate students' mathematical play while at the same time meaningfully leveraging their formal mathematics knowledge. We consider this kind of "productive" playful engagement to be an interaction between the task(s), student(s), and the teacher's implementation. We use the term *playful math* to describe the elements of classroom activities and environments that facilitate mathematical play. To investigate this broader question, we studied students engaging in playful math activities in an undergraduate class. Our research questions for this study are: (1) What features of the playful math environment supported or constrained students' productive use of their formal mathematics knowledge? (2) What features of the playful math environment supported or constrained students' mathematical play?

Background and Framing

Our two-part definition of mathematical play is grounded in the traits of a person's behavior and experience that are considered foundational in the literature. Firstly, mathematical play entails *agency in exploration* and *self-selection of goals or subgoals*. A person experiences freedom and control over their activity, including how they explore and the goals they select (Holton et al., 2001; Gresalfi et al., 2018). Exploration is essential (Mason, 2019) and often accompanied by unexpected outcomes (Davis, 1996). Goals are inherent to play (Huizinga,

1955) and can emerge and change as play proceeds (Jasien & Horn, 2022, Radke & Ma, 2018). Secondly, mathematical play entails *immersion* and *enjoyment*. People who play are highly engaged (Gresalfi et al., 2018) and may exhibit a state of flow (Csíkszentmihályi, 1990). They find pleasure in their activity and experience it as being worthwhile for its own sake (Su, 2017).

This definition highlights a tension that arises when mathematical play is used to support learning in classroom environments. Students with high agency might pursue goals that have little to do with the teacher's learning goals. Pyle and Danniels (2017) view play-based learning to be situated in a continuum, where the activity ranges from student-directed to teacher-directed. In more teacher-directed play, the broad goals can be prescribed if students have significant leeway in the subgoals and actions they can take to pursue them. This type of continuum provides a way to situate playful math activities at the undergraduate level also, where a teacher may direct students to engage with formal mathematics while still playing.

We have identified six design principles for playful math activities (see Ellis, et al., in press): (1) foster understanding of specific mathematical ideas; (2) enable free exploration within constraints; (3) engender anticipation within the task; (4) provide a method for intrinsic feedback; (5) offer meaningful challenge while still being feasible; and (6) allow the student to act as both designer and player. For this study, we considered what the students did with the intrinsic feedback they received from the task environment. A student experiences intrinsic feedback in direct correspondence to their actions, rather than as mediated by a teacher or some other extrinsic source. Students can increasingly come to anticipate the results of their actions in the environment when they are able to use their mathematical understanding to simulate and predict the experience first (Tzur, 2007). The feedback students receive can therefore inform their mental models and anticipated actions and outcomes, rather than just allowing for reactive adjustments to their actions. For example, Williams-Pierce et al. (2024) studied a case in which students programmed a robot to navigate a path and received perceptual feedback when the robot missed the mark. Because the students had developed and used a quantitative model while doing the programming, they anticipated the path and were able to interpret the feedback in light of their model, productively engaging with their prior mathematics knowledge.

Methods

We collected data with 16 secondary pre-service teachers (PSTs) at a public university in the southeast United States. The PSTs had taken at least two calculus courses and were enrolled in an algebraic reasoning content course. They formed 8 pairs, and each pair engaged in two playful activities on an iPad over four sessions. We collected audio and screen recordings of each pair's work as well as any written work not captured by the iPad. This report is based on the engagement of one PST pair, Audrey and Penny (pseudonyms), in one of these activities called *Major Tom*. *Major Tom* (Figure 1) consists of 6 tasks whose common objective is to move the astronaut ten miles "north" in space, from (0, 0) to (0, 10), in 10 minutes by adjusting "boosts" (accelerations). The PSTs entered magnitudes, angles, and durations of the boosts, hit Play, and the system showed the astronaut's motion. The system also provided quantitative feedback (time and fuel usage during the first session, position also during the second session). The activity's introduction read: "You are floating in space, at the origin. In 10 minutes, the spaceship will swing by the pick-up point ten miles 'north' of you. Can you be there at the right time? Your jetpack can give you a boost if you open its valve, in any direction you want. At each moment, the valve setting determines the acceleration you experience (in mi/min^2) at that moment. Change the table to make whatever sequence of valve settings you want."

On successive screens are follow-up tasks. Task 2 requires Tom to be stationary at the pick-up point. Task 3 introduces a fuel gauge (a valve setting of n for m seconds uses nm kg of fuel), and requires the use of specific amounts of fuel. In Task 4, an asteroid blocks the way, so Tom cannot stay on the y-axis. In Task 5, boosts can now include variables. In Task 6 you design your own challenges, which you solve in Task 7. *Major Tom* satisfies our design principles, allowing PSTs to select and explore their own goals, use feedback to reflect on their solution strategies, and design challenges. To save space we will use a shortened notation for inputs. For instance, the sequence of two boosts in Figure 1 would be written as {1, 90°, 0, 2} and {2, 180°, 4, 5}.

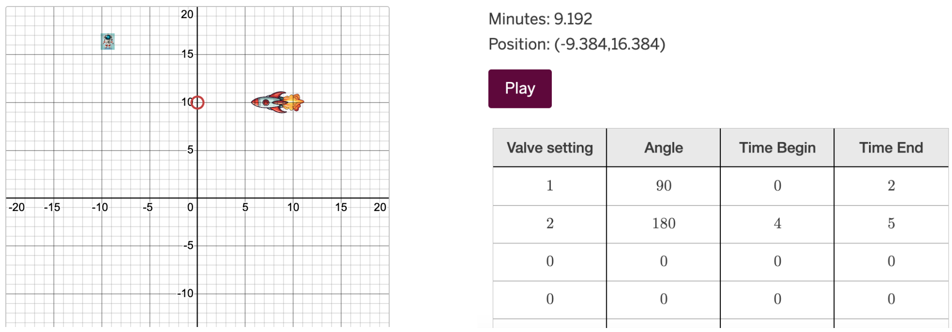

Figure 1. Major Tom Activity.

We intended the *Major Tom* activity to promote an understanding of the relationship between position, velocity, and acceleration in terms of derivatives and integrals. For instance, suppose a PST tried out one boost {1, 90°, 0, 2} followed by Tom coasting the rest of the way from $t = 2$ to 10. One can mathematically model Tom's velocity, whose derivative is the acceleration $a(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ 0, & 2 < t \leq 10 \end{cases}$, to get $v(t) = \begin{cases} t, & 0 \leq t \leq 2 \\ 2, & 2 \leq t \leq 10 \end{cases}$. One could then find the total displacement by integration, either by reasoning with areas under the velocity graph or symbolically as $\int_0^{10} v(t)dt = \frac{1}{2}t^2\big|_0^2 + 16 = 18$ miles. These models could support PSTs to get exact answers and examine more phenomena such as minimizing fuel usage or going in non-vertical directions. The PSTs engaged with *Major Tom* in two 75-minute sessions.

This is part of a larger study, so we adopted an existing coding scheme to describe the characteristics of mathematical play (Ellis et al., in press) as well as developed emergent codes to investigate the PSTs' cognitive engagement with a particular emphasis on their use of feedback. Part of the existing scheme relevant to this report concerns the characteristics of playful engagement, in which we distinguished moment phenomena from task phenomena. Moment phenomena are discrete instances of experience or behavior, and include *competitive fun*, *feeling proud*, *enjoyment*, *taking on authority*, and *laugh* (as evidence of enjoyment). Task phenomena reflect an outcome of the PSTs' engagement with a task as a whole, and include *investment*, *immersion*, *agency* in goal exploration or accomplishment, and *perseverance*. We coded the transcripts independently, discussing and resolving any disagreements to refine and establish a common understanding of the codes during weekly meetings. Once we developed a stable coding system, Authors 2 and 4 then re-coded the data corpus with the final system.

Results

Features influencing PST's productive use of their formal mathematics knowledge (RQ 1)

In the first session, the PSTs worked in pairs on Tasks 1-7. They did not always clearly distinguish between acceleration and velocity, and many had difficulties drawing on calculus concepts to try to produce exact solutions. At the beginning of the second session, the first author therefore led a structured discussion on constructing a velocity graph from a sequence of unidirectional accelerations (i.e., parallel to the y-axis) and discussed what the graph informs about displacement. The PSTs then revisited the tasks from the first session and solved new challenges they created. In examining what supported and constrained the PSTs' use of formal mathematics, we found two features. One is the imprecision of the program's feedback, which acted as a hindrance; the other is the structured discussion, which acted as a support.

A hindrance: Imprecise feedback from the system. The program's imprecise feedback hindered Audrey and Penny's productive use of their formal knowledge. We initially programmed *Major Tom* with a relaxed success condition: the spaceship would stop if Tom was less than 1 mile from the pick-up point (0, 10) at time $t = 10$ minutes, and would otherwise cruise past without stopping. Furthermore, the screen did not show a position readout for Tom. For the second session we added the position readout and shrunk the window of success to 0.1 mile. In this section we illustrate how this imprecise feedback in the first session diverted Audrey and Penny from drawing on the derivative/integral relationships between acceleration, velocity, and displacement to determine exact displacements for Tom.

On Task 1, Audrey and Penny started with the values seeded in the table and changed them one at a time for five tries, to understand the system. They then tried just one boost: {1, 90°, 0, 2}. When this took Tom too far, they tried {1, 90°, 0, 1}. With this, the spaceship stopped, with Tom at (0, 9.5) at time 10 (although the exact position was hard to discern). They considered this a success and moved on to Task 2, at no point using symbolic or numeric calculation. On Task 2, Audrey and Penny kept the same first line in their table, but added a second boost: {1, 90°, 0, 1}, {-1, 0°, 9, 10}. Tom veered at the end, so they corrected the angle from 0° to 90° and tried again. This time Tom stopped at (0, 9), short of the pick-up point. They kept making small adjustments, six different times, and eventually solved the problem with {1, 90°, 0, 1}, {-2, 90°, 9.5, 10}. The alert reader may calculate that this solution puts Tom at (0, 9.25) at $t = 10$, which is short of (0, 10), yet the program considered this close enough and the spaceship stopped.

Audrey and Penny continued to almost exclusively rely on this repeated guess-check-adjust method for the rest of the first session. They experienced success each time after multiple tries, never attaining an exact answer of (0, 10) at time 10. On Task 3 they used exactly 3kg of fuel with {1, 90°, 0, 1.5}, {-1, 90°, 1.5, 2}, {-2, 90°, 9.5, 10}, ending with Tom at (0, 9.75). On Task 4, with an asteroid in the way at (0, 5), they adapted their previous approaches to split the journey into two parts, and after many adjustments ended up with an approximation that got the spaceship to stop: {2, 45°, 0, 1}, {-2, 45°, 4, 5}, {2, 140°, 5, 6}, {-2, 140°, 8.8, 10}. They did not appeal to a mathematical model, never calculating anything with $\sqrt{2}$ or using a symmetric 135° angle to try for an exact solution. On Task 5, they were asked to use some variable non-constant valve settings (and still avoiding the asteroid at (0, 5)). The aim was to push the PSTs to reason with integrals to calculate displacements when the accelerations were non-constant (and thus required calculus). Penny did briefly start by writing some calculus symbolism, "ds^2/d^2t," but she quickly abandoned this approach and instead decided to adapt their solution from the previous task, inputting {$4x$, 45°, 0, 1}, {$-4(x-4)$, 45°, 4, 5}, {$4(x-5)$, 140°, 5, 6}, {$-4(x-9)$, 140°, 9, 10}. She explained to Audrey, "We want our average acceleration for one minute to be 1, which means it should be $2x$." Thus, an acceleration of $4x$ over the time interval $x = 0$ to 1 would average to 2, so she assumed that replacing every 2 with $4x$ would not affect the displacement.

Penny conflated some aspects of velocity with acceleration here. If the *velocity* were $4x$, it could be averaged to 2 over this time interval without changing the total displacement, but the same is not true when the *acceleration* is $4x$. The displacement s over one minute is 2/3 when

$\frac{d^2s}{dx^2} = 4x$ and 1 when $\frac{d^2s}{dx^2} = 2$. This difference, accumulated over their four small time intervals, put Tom a bit beyond (0, 10) when the spaceship came by. But because Major Tom was within 1 mile of the target, it stopped anyway. Audrey and Penny made further adjustments to get Tom visibly closer to (0, 10), but, like before, they moved to the next task without getting an exact answer. Getting an exact answer was unimportant to them, as it was not necessary to make the spaceship stop. This laxity of feedback thus allowed Audrey and Penny to avoid confronting a conceptual issue about the relationship between acceleration, velocity, and displacement.

A support: Structured discussion. During the beginning of the second session, the PSTs discussed how the area under the curve (i.e., between the graph and the time axis) represented the total distance traveled. This discussion supported Audrey and Penny's productive use of their formal mathematics knowledge. They proceeded to engage with the activity very differently when they revisited the tasks from the previous session, including the challenges they had invented. They no longer conflated acceleration and velocity. For each task they drew a velocity graph, and they consistently interpreted the accelerations as the slopes of this graph, recognizing acceleration to be the derivative of velocity. They interpreted Tom's total displacement as the definite integral of the graph. An example is in Figure 2, which shows Penny's mathematical model of her solution to Task 2, where Tom must be stationary at (0, 10) at $t = 10$. This allowed them to calculate inputs to yield an exact solution which they could then test with the system, instead of repeatedly guessing, checking, and adjusting as they did in the first session.

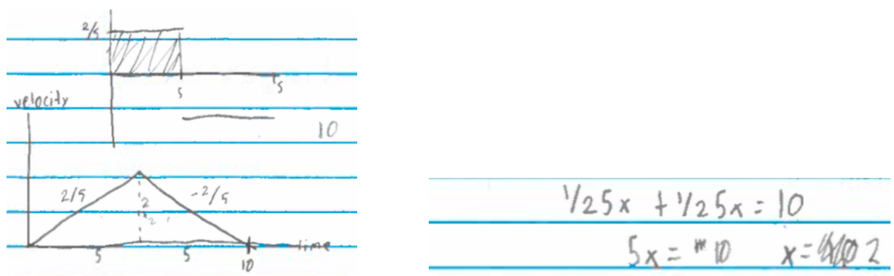

Figure 2. Penny's velocity graph model for solving Task 2, second session: {2/5, 90°, 0, 5}, {-2/5, 90°, 5, 10}.

Velocity graphing also helped the PSTs use their calculus knowledge to create new scenarios. For instance, they invented this challenge: "Major Tom needs to go to (0, 15) to pick up some space litter but still needs to get to the rocket. Figure out how he can do this and be stationary at the pick-up point at least one minute early." They used a velocity graph with negative area (Figure 3) to calculate a solution: {5/3, 90°, 0, 3}, {-5/3, 90°, 3, 6}, {-20/9, 90°, 6, 7.5}, {20/9, 90°, 7.5, 9}. Penny initially calculated a solution that she found to be incorrect when she checked it on *Major Tom.* Seeing Tom miss the mark, she said, "Oh wait, no, no, no, I see what I did. I see what I did. So, I made the area of this [the negative region] ten. It's supposed to be five." Then she said, "This is why I like having this [the program] to check. I wouldn't have realized I did it wrong." Thus, the velocity graph model not only helped them solve the problem, but also helped them interpret and resolve perturbing feedback from the system.

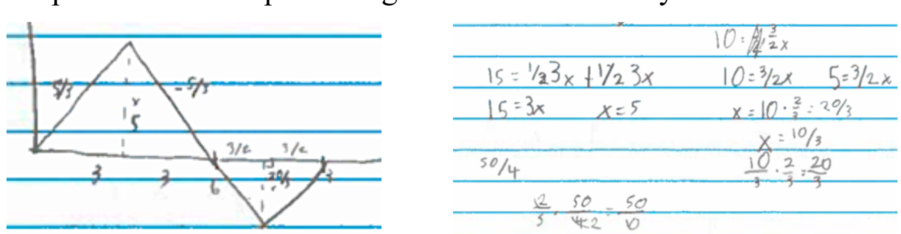

Figure 3. Penny's model for solving their invented challenge, second session.

Features Influencing Mathematical Play (RQ 2)

To investigate the features of the environment that may have influenced mathematical play, we considered the incidence of our moment and task codes. *Competitive fun* occurred when the PSTs demonstrated excitement about their challenges, *feeling proud* occurred when they showed pride in their work, *enjoyment* occurred when they demonstrated enthusiasm (further evidence of this was through the *laugh* code), and we coded *authority* when seeing either PST take on that role. We coded for *investment* when the PSTs showed that they cared about their work or the task's outcome. Evidence for *immersion* included the PSTs ignoring distractions and lacking awareness of time passing. *Agency* occurred when the PSTs demonstrated autonomy in choosing a strategy or posing their own problem, and we coded for *perseverance* when we found evidence of the PSTs voluntarily persisting through difficulty.

Audrey and Penny did demonstrate evidence of mathematical play. Across both days they engaged in a total of 14 tasks, spending on average eight and a half minutes per task (although there was great variation across the tasks). Considering the moment phenomena, competitive fun occurred 0.8 times per task on average, feeling proud 1.6 times, enjoyment 1.1 times, taking on authority 0.6 times, and laugh occurred 6.2 times. The task phenomena codes we either applied zero or one time per task. Out of the 14 tasks, we found evidence for investment during 8 of the tasks, immersion during 2 tasks, agency during 10 tasks, and perseverance during 5 tasks. In considering whether the imprecise feedback influenced Audrey and Penny's mathematical play, we noted that it was a feature of every task, and thus the presence of mathematical play phenomena suggest that it did not prevent them from playful engagement. However, some aspects of mathematical play were minimal, such as immersion, taking on authority, and experiencing a sense of competitive fun. To examine the potential effect of the structured discussion, we compared the incidence of mathematical play prior to and after the discussion. Audrey and Penny solved 8 tasks prior to the discussion and 6 tasks after. The length of time they spent per task was slightly more prior (9.2 minutes) than after (7.4 minutes). If the structured discussion affected the PSTs' playful engagement, we would expect to see a noticeable difference in the average incidence per task prior to the discussion compared to after. Except for the laugh code, we did not see any meaningful difference in the moment phenomena. Two phenomena decreased slightly (competitive fun and feeling proud), and two increased slightly (enjoyment and authority). Laughter, however, did decrease noticeably, occurring 7.4 times per task on average prior to the structured discussion and only 4.7 times per task on average after. The students made more verbal expressions of enjoyment after the discussion, but they laughed less. Considering the task phenomena, we saw a slight decrease in the incidence of investment and perseverance and a slight increase in the incidence of immersion. Agency, however, doubled, occurring for only half the tasks prior to the structured discussion and for every task after. It may be that Audrey and Penny experienced greater agency having gained an appreciation for the utility of velocity graphing, derivative, and integral to tackle the challenges.

We then returned to the data to see whether there were any features of the task / instructional environment that *did* influence the PSTs' playful engagement. We found one feature, which was the creation and solution of their own challenges. Audrey and Penny solved nine regular tasks (in which they solved a problem given to them), two "make a challenge" tasks, and three "solve your own challenge" tasks. Figure 4 shows the average number of moment and task phenomena per task for the regular tasks compared to the make-a-challenge and solve-your-challenge tasks. One caveat is that the task time differed significantly, which would affect the incidence of moment codes (but not task codes). Audrey and Penny spent longer on average on the regular tasks compared to the make a challenge tasks (5.9 minutes versus 3.3 minutes), and they spent a whopping 19.4 minutes on average when solving their own challenges. Thus, there was more time to observe and code moment phenomena during the solve your challenge tasks. This accounts for the large discrepancy of laugh codes, for instance. However, it is interesting to observe that the PSTs laughed 7.5 times per task when making a challenge compared to 2.3 times per regular task, despite make-a-challenge tasks being shorter. Moreover, when we consider task phenomena, which are not swayed by task length, we still saw greater levels of investment, immersion, and perseverance when solving their own challenges. It appears that having the agency to design and solve problems of their choice was meaningful for Audrey and Penny, and it enabled them to take ownership in their work and invest in the outcome of their activity.

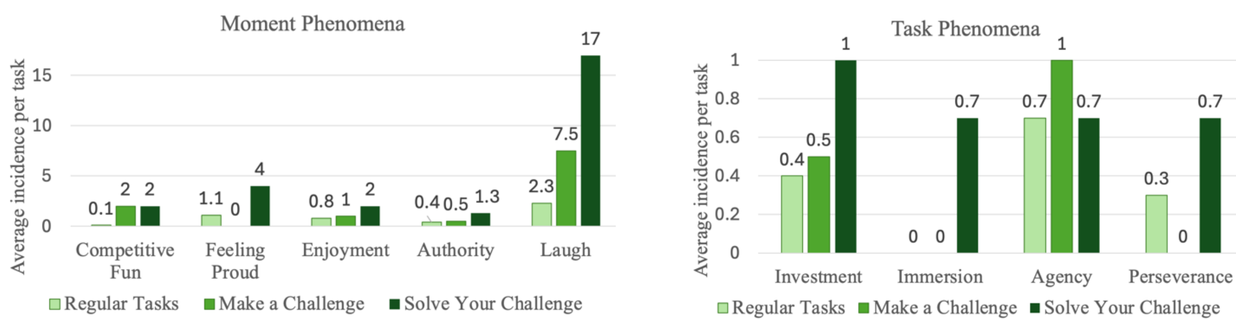

Figure 4. Comparison of mathematical play phenomena across task types.

Discussion

Our findings with Audrey and Penny playing with the *Major Tom* activity provide detail that can augment our design principles for playful math tasks. One of those principles is to provide a method for intrinsic feedback. This case indicates that it is worth considering whether this feedback is also precise and quantitative enough to be reasonably guess-and-check-proof. This can help to necessitate (or at least to not allow the students to easily avoid) the engagement of formal math knowledge. The feedback can then bear on their mental models, not just on their immediate actions. The design factor of allowing students to act as both designer and player was certainly reinforced for these students, for whom inventing and solving their own challenges was propitious for play. Our findings also reinforce the design factor of fostering understanding of specific mathematical ideas. In particular, it illustrates that there need not be a trade-off between play and engagement with formal mathematics; the teacher-directed discussion supported the latter without diminishing the former or dampening student agency.

As we reported on just one pair of students, we do not wish to make any general claims from them, but rather to note that they reveal how various features of a playful math environment can function. Since other students may be hindered and helped by different factors, based on their dispositions and knowledge, we aim to continue studying this question in the context of other playful activities in other undergraduate mathematics classrooms.

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